# Overview

In preparation for the exam, we'll look at the questions asked on the 2013 Mid-Semester Exam.

# Sample Question: Lines & Planes

Let $P$ be the plane in $\mathbb{R}^3$ defined by the equation $2x + y - z = 1$, and let $L$ be the line through the point $(1, 1, 1)$ which is orthogonal to $P$.

1. Find an equation for $P$ of the form $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ for some vector $\mathbf{n}$ and some vector $\mathbf{r_0}$.
2. Find an equation for $L$.
3. Let $Q$ be the plane containing $L$ and the point $(1, 1, 2)$. Find an equation for $Q$.

# Solution: Lines & Planes

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1. Find an equation for $P$ of the form $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ for some vector $\mathbf{n}$ and some vector $\mathbf{r_0}$.

To find the equation of a plane $P$, we need a **normal vector** to $P$ and a **point** on $P$.

The plane $Ax + By + Cz + D = 0$ has normal vector $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$, so a normal vector to $P$ is given by $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. To find a point on $P$, we can plug in $x = y = 0$ and see that $(0, 0, -1)$ satisfies the equation $2x + y - z = 1$. Thus the general formula $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ becomes

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} = 0.$$

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2. Find an equation for $L$.

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2. Find an equation for $L$.

A direction vector for $L$ is any normal vector to $P$: i.e., any scalar multiple of $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. This yields the vector equation

$$\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix},$$

with the associated parametric equations

$$x = 1 + 2t \quad y = 1 + t \quad z = 1 - t.$$

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3. Let $Q$ be the plane containing $L$ and the point $(1, 1, 2)$. Find an equation for $Q$.

To find a normal vector to the new plane, take the cross product of two vectors parallel to $Q$. For example, you could choose a direction vector for $L$ and the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ between the two given points on $Q$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j}.$$

Any equation for the plane is acceptable, including the following:

$$\left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = 0,$$

# Sample Question: Bases & Coordinates

The set $\mathcal{B} = \{t+1, 1+t^2, 3-t^2\}$ is a basis for $\mathbb{P}_2$.

1. If $p(t) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{B}}$, express $p$ in the form $p(t) = a + bt + ct^2$.
2. Find the coordinate vector of the polynomial $q(t) = 2 - 2t$ with respect to $\mathcal{B}$ coordinates.

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Since the $\mathcal{B}$ coordinates of $p$ are $1, 1$, and $-1$, we have

$$p(t) = 1(t+1) + 1(1+t^2) - 1(3-t^2) = -1 + t + 2t^2.$$

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The set $\mathcal{B} = \{t+1, 1+t^2, 3-t^2\}$ is a basis for $\mathbb{P}_2$.

2. Find the coordinate vector of the polynomial $q(t) = 2 - 2t$ with respect to $\mathcal{B}$ coordinates.

We need $a, b$, and $c$ such that

$$a(t+1) + b(1+t^2) + c(3-t^2) = 2 - 2t.$$

Collecting like powers of $t$ gives us a system of equations:

$$\begin{aligned} a + b + 3c &= 2 \\ a &= -2 \\ b - c &= 0. \end{aligned}$$

The unique solution to this is $a = -2$, $b = c = 1$.
To protect against algebra mistakes, check that

$$-2(t+1) + 1(1+t^2) + 1(3-t^2) = 2 - 2t.$$

# Sample Question: Vector Spaces

Decide whether each of the following sets is a vector space. If it is a vector space, state its dimension. If it is not a vector space, explain why.

1. $A$ is the set of $2 \times 2$ matrices whose entries are integers.
2. $B$ is the set of vectors in $\mathbb{R}^3$ which are orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.
3. $C$ is the set of polynomials whose derivative is 0:

$$C = \{p(x) \in \mathbb{P} \mid \frac{d}{dx} p(x) = 0\}.$$

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1. $A$ is the set of $2 \times 2$ matrices whose entries are integers.

This is a subset of the vector space of $2 \times 2$ matrices with real entries, so we can check if the three subspace axioms hold:

1. Is 0 in the set?
2. Is the set closed under addition?
3. Is the set closed under scalar multiplication?

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3. Is the set closed under scalar multiplication?

No, this is not a vector space. This set is not closed under multiplication by a non-integer scalar. For example,

$$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \text{ is not in } A.$$

# Solution: Vector Spaces

Decide whether each of the following sets is a vector space. If it is a vector space, state its dimension. If it is not a vector space, explain why.

2. $B$ is the set of vectors in $\mathbb{R}^3$ which are orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

As before, we could check the 3 subspace axioms, but it's quicker to observe that $B$ is the null space of the matrix $[1 \;\; 0 \;\; 2]$, and the null space of a matrix is always a subspace.
We can find a basis for the null space explicitly and check that it has 2 vectors. Alternatively, observe that the matrix $[1 \;\; 0 \;\; 2]$ has rank 1, so its null space is two-dimensional by the Rank Theorem.

# Checking the 3 subspace axioms

1. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0$, so $\mathbf{0} \in B$.
2. Suppose $\mathbf{v}, \mathbf{u} \in B$. Then $\mathbf{v} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \mathbf{u} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0.$
$$(\mathbf{u}+\mathbf{v}) \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \mathbf{u} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \mathbf{v} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0 + 0 = 0.$$
Since $\mathbf{u}+\mathbf{v}$ is in $B$, $B$ is closed under addition.
3. Suppose $\mathbf{v} \in B$.
$$(c\mathbf{v}) \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = c\left(\mathbf{v} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) = c0 = 0.$$
Since $c\mathbf{v}$ is in $B$, $B$ is closed under scalar multiplication.

# Solution: Vector Spaces

Decide whether each of the following sets is a vector space. If it is a vector space, state its dimension. If it is not a vector space, explain why.

3. the set of polynomials whose derivative is 0:

$$C = \left\{ p(x) \in \mathbb{P} \;\middle|\; \frac{d}{dx} p(x) = 0 \right\}.$$

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We can solve this problem by recognising that the polynomials whose derivatives are 0 are exactly the constant polynomials, so $C = \mathbb{R}^1$. It follows that $C$ is a one-dimensional vector space.

It is also acceptable to show that $C$ is a subspace of the vector space $\mathbb{P}$ by verifying each of the subspace axioms.

# Sample Question: Linear transformations

A linear transformation $T : M_{2\times 2} \to M_{2\times 2}$ is defined by:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

(a) Calculate $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.

(b) Which, if any, of the following matrices are in $\ker(T)$?

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

(c) Which, if any, of the following matrices are in $\operatorname{range}(T)$?

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Find the kernel of $T$ and explain why $T$ is not one to one.

(e) Explain why $T$ does not map $M_{2\times 2}$ onto $M_{2\times 2}$.

# Sample Question: Subspaces associated to a matrix

Consider the matrix $A$:

$$\begin{bmatrix} 2 & -4 & 0 & 2 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 1 & 4 \end{bmatrix}.$$

(i) Find a basis for $\operatorname{Nul} A$.

(ii) Find a basis for $\operatorname{Col} A$.

(iii) Consider the linear transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$. Give a geometric description of the range of $T_A$ as a subspace of $\mathbb{R}^3$. What is its dimension? Does it pass through the origin?

We begin by row-reducing $A$:

$$\begin{bmatrix} 2 & -4 & 0 & 2 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 1 & 4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Find a basis for $\operatorname{Nul} A$.

The general solution to $R \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = 0$ is $y + 3z = 0$, $w - 2x + z = 0$, so

$$\operatorname{Nul} A = \left\{ \begin{bmatrix} 2x - z \\ x \\ -3z \\ z \end{bmatrix} \right\} = \left\{ x \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

and so $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for $\operatorname{Nul} A$.

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(ii) Find a basis for Col $A$.

A basis for Col $A$ is obtained by taking every column of $A$ that corresponds to a pivot column in the row reduced form of $A$. Thus the first and third columns

$$\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

form a basis for Col $A$.

(iii) Consider the linear transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$. Give a geometric description of the range of $T_A$ as a subspace of $\mathbb{R}^3$. What is its dimension? Does it pass through the origin?

The range of $T_A$ is exactly the column space of $A$. We just saw that it has a basis with two elements, so it is two dimensional. It is a plane in $\mathbb{R}^3$, and passed through the origin, because every vector subspace contains $\mathbf{O}$.

# Revision: Definitions

- What is a vector space? Give some examples.
- What is a subspace? How do you check if a subset of a vector space is a subspace?
- What is a linear transformation? Give some examples.
- What does it mean for a set of vectors to be linearly independent? How do you check this?
- What are the coordinates of a vector with respect to a basis?

# Revision: Geometry of $\mathbb{R}^3$

- What information do you need to determine a line? A plane?
- How can you check if two lines are orthogonal? Parallel?
- How do you find the distance between a point and a line? A point and a plane?
- How can you find the angle between two vectors?
- What are the scalar and vector projections of one vector onto another? Can you describe these in words?

# Revision: Bases

- What is a basis for a vector space?
- If the dimension of $V$ is $n$, then $V$ and $\mathbb{R}^n$ are *isomorphic*. What does this mean and how do we know it's true?
- In an $n$-dimensional vector space,
  - any $n$ linearly independent vectors form a basis.
  - any $n$ vectors which span $V$ form a basis.
  - any set of vectors which spans $V$ contains a basis for $V$.
  - any set of linearly independent vectors can be extended to a basis for $V$.
- How do you find a basis for the null space of a matrix? The column space? The row space? The kernel of the associated linear transformation?

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  - any set of vectors which spans $V$ contains a basis for $V$.
  - any set of linearly independent vectors can be extended to a basis for $V$.
- How do you find a basis for the null space of a matrix? The column space? The row space? The kernel of the associated linear transformation? (Which pair of these are the same?)