Overview

Before the break, we began to study eigenvectors and eigenvalues, introducing the characteristic equation as a tool for finding the eigenvalues of a matrix:

$$\det(A - \lambda I) = 0.$$

The roots of the characteristic equation are the eigenvalues of λ.
We also discussed the notion of similarity: the matrices A and B are *similar* if $A = PBP^{-1}$ for some invertible matrix P.

Question

When is a matrix A similar to a diagonal matrix?

From Lay, §5.3

Quick review

Definition

An *eigenvector* of an $n \times n$ matrix A is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is an *eigenvalue* for A.

To find the eigenvalues of a matrix, find the solutions of the characteristic equation:

$$\det(A - \lambda I) = \mathbf{0}.$$

The λ-eigenspace is the set of all eigenvectors for the eigenvalue λ, together with the zero vector. The λ-eigenspace E_λ is Nul $(A - \lambda I)$.

The advantages of a diagonal matrix

Given a diagonal matrix, it's easy to answer the following questions:

1. What are the eigenvalues of D? The dimensions of each eigenspace?
2. What is the determinant of D?
3. Is D invertible?
4. What is the characteristic polynomial of D?
5. What is D^k for $k = 1, 2, 3, \ldots$?

The advantages of a diagonal matrix

Given a diagonal matrix, it's easy to answer the following questions:

1. What are the eigenvalues of D? The dimensions of each eigenspace?
2. What is the determinant of D?
3. Is D invertible?
4. What is the characteristic polynomial of D?
5. What is D^k for $k = 1, 2, 3, \ldots$?

For example, let $D = \begin{bmatrix} 10^{50} & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & -2.7 \end{bmatrix}$.

Can you answer each of the questions above?

The diagonalisation theorem

The goal in this section is to develop a useful factorisation $A = PDP^{-1}$, for an $n \times n$ matrix A. This factorisation has several advantages:

- it makes transparent the geometric action of the associated linear transformation, and
- it permits easy calculation of A^k for large values of k:

Example 1

Let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Then the transformation T_D scales the three standard basis vectors by $2, -4$, and -1, respectively.

$$D^7 = \begin{bmatrix} 2^7 & 0 & 0 \\ 0 & (-4)^7 & 0 \\ 0 & 0 & (-1)^7 \end{bmatrix}.$$

Example 2

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. We will use similarity to find a formula for A^k. Suppose we're given $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$.

We have

$$\begin{aligned} A &= PDP^{-1} \\ A^2 &= PDP^{-1}PDP^{-1} \end{aligned}$$

Example 2

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. We will use similarity to find a formula for A^k. Suppose we're given $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$.

We have

$$\begin{aligned} A &= PDP^{-1} \\ A^2 &= PDP^{-1}PDP^{-1} \\ &= PD^2P^{-1} \\ A^3 &= PD^2P^{-1}PDP^{-1} \end{aligned}$$

Example 2

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. We will use similarity to find a formula for A^k. Suppose we're given $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$.

We have

$$\begin{aligned} A &= PDP^{-1} \\ A^2 &= PDP^{-1}PDP^{-1} \\ &= PD^2P^{-1} \\ A^3 &= PD^2P^{-1}PDP^{-1} \\ &= PD^3P^{-1} \end{aligned}$$

Example 2

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. We will use similarity to find a formula for A^k. Suppose we're given $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$.

We have

$$\begin{array}{rcl} A & = & PDP^{-1} \\ A^2 & = & PDP^{-1}PDP^{-1} \\ & = & PD^2P^{-1} \\ A^3 & = & PD^2P^{-1}PDP^{-1} \\ & = & PD^3P^{-1} \\ \vdots & \vdots & \vdots \\ A^k & = & PD^kP^{-1} \end{array}$$

So

$$A^k = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 4^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 2/5 & 3/5 \\ 1/5 & -1/5 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2}{5}4^k + \frac{3}{5}(-1)^k & \frac{3}{5}4^k - \frac{3}{5}(-1)^k \\ \frac{2}{5}4^k - \frac{2}{5}(-1)^k & \frac{3}{5}4^k + \frac{2}{5}(-1)^k \end{bmatrix}$$

Diagonalisable Matrices

Definition

An $n \times n$ (square) matrix is **diagonalisable** if there is a diagonal matrix D such that A is similar to D.

That is, A is diagonalisable if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$ (or equivalently $A = PDP^{-1}$).

Diagonalisable Matrices

Definition

An $n \times n$ (square) matrix is **diagonalisable** if there is a diagonal matrix D such that A is similar to D.

That is, A is diagonalisable if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$ (or equivalently $A = PDP^{-1}$).

Question

How can we tell when A is diagonalisable?

The answer lies in examining the eigenvalues and eigenvectors of A.

Recall that in Example 2 we had

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \text{ and } A = PDP^{-1}.$$

Note that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$A \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = -1 \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Recall that in Example 2 we had

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \text{ and } A = PDP^{-1}.$$

Note that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$A \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = -1 \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

We see that each column of the matrix P is an eigenvector of A...

This means that we can view P as a change of basis matrix from eigenvector coordinates to standard coordinates!

In general, if $AP = PD$, then

$$A \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

If $\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$ is invertible, then A is the same as

$$\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}^{-1}.$$