Overview

Yesterday we studied how real 2×2 matrices act on $\mathbb{C}$. Just as the action of a diagonal matrix on $\mathbb{R}^2$ is easy to understand (i.e., scaling each of the basis vectors by the corresponding diagonal entry), the action of a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ determines a composition of rotation and scaling.
We also saw that any 2×2 matrix with complex eigenvalues is similar to such a "standard" form.
Today we'll return to the study of matrices with real eigenvalues, using them to model discrete dynamical systems.

From Lay, §5.6

The main ideas

In this section we will look at discrete linear dynamical systems. *Dynamics* describe the evolution of a system over time, and a *discrete* system is one where we sample the state of the system at intervals of time, as opposed to studying its continuous behaviour. Finally, these systems are *linear* because the change from one state to another is described by a vector equation like

$$(*) \qquad \mathbf{x}_{k+1} = A\mathbf{x}_k\ .$$

where A is an $n \times n$ matrix and the $\mathbf{x}_k$'s are vectors $\mathbb{R}^n$.

You should look at the equation above as a recursive relation. Given an initial vector $\mathbf{x}_0$ we obtain a sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, ..$ where for every k the vector $\mathbf{x}_{k+1}$ is obtained from the previous vector $\mathbf{x}_k$ using the relation $(*)$. We are generally interested in the long term behaviour of such a system.

The applications in Lay focus on ecological problems, but also apply to problems in physics, engineering and many other scientific fields.

Initial assumptions
We'll start by describing the circumstances under which our techniques will be effective:

- The matrix A is diagonalisable.
- A has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues
$\lambda_1, \ldots, \lambda_n$.
- The eigenvectors are arranged so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$, any initial vector $\mathbf{x}_0$ can be written

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

This eigenvector decomposition of $\mathbf{x}_0$ determines what happens to the sequence $\{\mathbf{x}_k\}$.

Since

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n,$$

we have

$$\begin{aligned}
\mathbf{x}_1 = A\mathbf{x}_0 &= c_1 A\mathbf{v}_1 + \cdots + c_n A\mathbf{v}_n \\
&= c_1\lambda_1\mathbf{v}_1 + \cdots + c_n\lambda_n\mathbf{v}_n \\
\mathbf{x}_2 = A\mathbf{x}_1 &= c_1\lambda_1 A\mathbf{v}_1 + \cdots + c_n\lambda_n A\mathbf{v}_n \\
&= c_1(\lambda_1)^2\mathbf{v}_1 + \cdots + c_n(\lambda_n)^2\mathbf{v}_n
\end{aligned}$$

and in general,

$$\mathbf{x}_k = c_1(\lambda_1)^k\mathbf{v}_1 + \cdots + c_n(\lambda_n)^k\mathbf{v}_n \qquad (1)$$

We are interested in what happens as $k \to \infty$.

Predator - Prey Systems

Example

See Example 1, Section 5.6

The owl and wood rat populations at time k are described by $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, where k is the time in months, O_k is the number of owls in the region studied, and R_k is the number of rats (measured in thousands). Since owls eat rats, we should expect the population of each species to affect the future population of the other one.

The changes in theses populations can be described by the equations:

$$\begin{aligned}
O_{k+1} &= (0.5)O_k + (0.4)R_k \\
R_{k+1} &= -p \cdot O_k + (1.1)R_k
\end{aligned}$$

where p is a positive parameter to be specified.

In matrix form this is

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0.5 & 0.4 \\ -p & 1.1 \end{bmatrix} \mathbf{x}_k.$$

Example (Case 1)

$p = 0.104$

This gives $A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$

According to the book, the eigenvalues for A are $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$. Corresponding eigenvectors are, for example,

$$\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

An initial population $\mathbf{x}_0$ can be written as $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then for $k \geq 0$,

$$\begin{aligned} \mathbf{x}_k &= c_1(1.02)^k\mathbf{v}_1 + c_2(0.58)^k\mathbf{v}_2 \\ &= c_1(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2(0.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix} \end{aligned}$$

As $k \to \infty$, $(0.58)^k \to 0$. Assume $c_1 > 0$. Then for large k,

$$x_k \approx c_1(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix}$$

and

$$\mathbf{x}_{k+1} \approx c_1(1.02)^{k+1} \begin{bmatrix} 10 \\ 13 \end{bmatrix} \approx 1.02\mathbf{x}_k.$$

The last approximation says that eventually both the population of rats and the population of owls grow by a factor of almost 1.02 per month, a 2% growth rate.

The ratio 10 to 13 of the entries in $\mathbf{x}_k$ remain the same, so for every 10 owls there are 13 thousand rats.

This example illustrates some general facts about a dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ when

- $|\lambda_1| \geq 1$ and
- $1 > |\lambda_j|$ for $j \geq 2$ and
- $\mathbf{v}_1$ is an eigenvector associated with λ_1.

If $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, with $c_1 \neq 0$, then for all sufficiently large k,

$$\mathbf{x}_{k+1} \approx \lambda_1\mathbf{x}_k \quad \text{and} \quad \mathbf{x}_k \approx c_1(\lambda)^k\mathbf{v}_1.$$

Example (Case 2)

We consider the same system when $p = 0.2$ (so the predation rate is higher than in the previous Example (1), where we had taken $p = 0.104 < 0.2$). In this case the matrix A is

$$\begin{bmatrix} 0.5 & 0.4 \\ -0.2 & 1.1 \end{bmatrix}.$$

Here

$$A - \lambda I = \begin{bmatrix} 0.5-\lambda & 0.4 \\ -0.2 & 1.1-\lambda \end{bmatrix}$$

and the characteristic equation is

$$\begin{aligned} 0 &= (0.5-\lambda)(1.1-\lambda) + (0.4)(0.2) \\ &= 0.55 - 1.6\lambda + \lambda^2 + 0.08 \\ &= \lambda^2 - 1.6\lambda + 0.63 \\ &= (\lambda - 0.9)(\lambda - 0.7) \end{aligned}$$

When $\lambda = 0.9$,

$$E_{0.9} = \text{Nul} \begin{bmatrix} -0.4 & 0.4 \\ -0.2 & 0.2 \end{bmatrix} \to \text{Nul} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

and an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

When $\lambda = 0.7$

$$E_{0.7} = \text{Nul} \begin{bmatrix} -0.2 & 0.4 \\ -0.2 & 0.4 \end{bmatrix} \to \text{Nul} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

and an eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

This gives

$$\mathbf{x}_k = c_1(0.9)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(0.7)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} \to \mathbf{0},$$

as $k \to \infty$.

The higher predation rate cuts down the owls' food supply, and in the long term both populations die out.

Example (Case 3)

We consider the same system again when $p = 0.125$. In this case the matrix A is

$$\begin{bmatrix} 0.5 & 0.4 \\ -0.125 & 1.1 \end{bmatrix}.$$

Hence

$$A - \lambda I = \begin{bmatrix} 0.5 - \lambda & 0.4 \\ -0.125 & 1.1 - \lambda \end{bmatrix}$$

and the characteristic equation is

$$\begin{aligned} 0 &= (0.5 - \lambda)(1.1 - \lambda) + (0.4)(0.125) \\ &= 0.55 - 1.6\lambda + \lambda^2 + 0.05 \\ &= \lambda^2 - 1.6\lambda + 0.6 \\ &= (\lambda - 1)(\lambda - 0.6). \end{aligned}$$

When $\lambda = 1$,

$$E_1 = \text{Nul} \begin{bmatrix} -0.5 & 0.4 \\ -0.125 & 0.1 \end{bmatrix} \to \text{Nul} \begin{bmatrix} 1 & -0.8 \\ 0 & 0 \end{bmatrix}$$

and an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}$.

When $\lambda = 0.6$

$$E_{0.6} = \text{Nul} \begin{bmatrix} -0.1 & 0.4 \\ -0.125 & 0.5 \end{bmatrix} \to \text{Nul} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

and an eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

This gives

$$\mathbf{x}_k = c_1(1)^k \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} + c_2(0.6)^k \begin{bmatrix} 4 \\ 1 \end{bmatrix} \to c_1 \begin{bmatrix} 0.8 \\ 1 \end{bmatrix},$$

as $k \to \infty$.

In this case the population reaches an equilibrium, where for every 8 owls there are 10 thousand rats. The size of the population depends only on the values of c_1.

This equilibrium is not considered stable as small changes in the birth rates or the predation rate can change the situation.

Graphical Description of Solutions

When A is a 2×2 matrix we can describe the evolution of a dynamical system geometrically.

The equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ determines an infinite collection of equations. Beginning with an initial vector $\mathbf{x}_0$, we have

$$\begin{aligned} \mathbf{x}_1 &= A\mathbf{x}_0 \\ \mathbf{x}_2 &= A\mathbf{x}_1 \\ \mathbf{x}_3 &= A\mathbf{x}_2 \\ &\vdots \end{aligned}$$

The set $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots\}$ is called a trajectory of the system.

Note that $\mathbf{x}_k = A^k\mathbf{x}_0$.

Examples

Example 1

Let $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}$. Plot the first five points in the trajectories with the following initial vectors:

$$(a)\ \mathbf{x}_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \quad (b)\ \mathbf{x}_0 = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

$$(c)\ \mathbf{x}_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad (d)\ \mathbf{x}_0 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Notice that since A is already diagonal, the computations are much easier!

(a) For $\mathbf{x}_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}$, we compute

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix} \qquad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 1.25 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 0.625 \\ 0 \end{bmatrix} \qquad \mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} 0.3125 \\ 0 \end{bmatrix}$$

These points converge to the origin along the x-axis.
(Note that $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for the matrix).

(b) The situation is similar for the case $\mathbf{x}_0 = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$, except that the convergence is along the y-axis.

(c) For the case $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, we get

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 2 \\ 3.2 \end{bmatrix} \qquad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2.56 \end{bmatrix}$$

$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 0.5 \\ 2.048 \end{bmatrix} \qquad \mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} 0.25 \\ 1.6384 \end{bmatrix}$$

These points also converge to the origin, but not along a direct line. The trajectory is an arc that gets closer to the y-axis as it converges to the origin.

The situation is similar for case (d) with convergence also toward the y-axis.

In this example every trajectory converges to $\mathbf{0}$. The origin is called an attractor for the system.

We can understand why this happens when we consider the eigenvalues of A: 0.8 and 0.5. These have corresponding eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
So, for an initial vector

$$\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we have

$$\mathbf{x}_k = A^k \mathbf{x}_0 = c_1 (0.8)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 (0.5)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Because both $(0.8)^k$ and $(0.5)^k$ approach zero as k gets large, $\mathbf{x}_k$ approaches $\mathbf{0}$ for any initial vector $\mathbf{x}_0$.
Because $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to the larger eigenvalue of A, $\mathbf{x}_k$ approaches a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as long as $c_1 \neq 0$.

Graphical example

Dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} .80 & 0 \\ 0 & .64 \end{bmatrix}$$

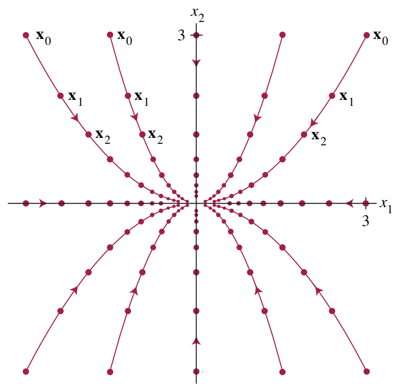


FIGURE 1 The origin as an attractor.

Example 2

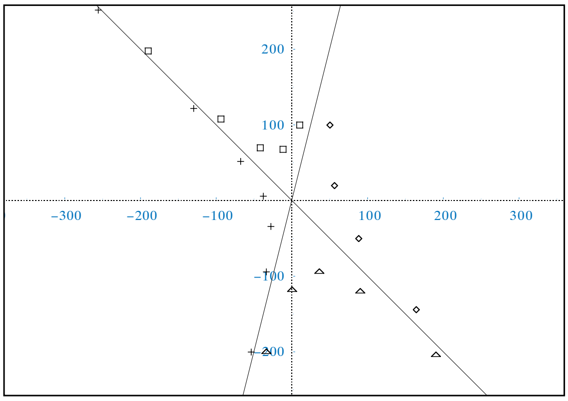
Describe the trajectories of the dynamical system associated to the matrix $A = \begin{bmatrix} 1.7 & -0.3 \\ -1.2 & 0.8 \end{bmatrix}$.

The eigenvalues of A are 2 and 0.5, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.
As above, the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ has solution

$$\mathbf{x}_k = 2^k c_1 \mathbf{v}_1 + (.05)^k c_2 \mathbf{v}_2$$

where c_1, c_2 are determined by $\mathbf{x}_0$.
Thus for $\mathbf{x}_0 = \mathbf{v}_1$, $\mathbf{x}_k = 2^k \mathbf{v}_1$, and this is unbounded for large k, whereas for $\mathbf{x}_0 = \mathbf{v}_2$, $\mathbf{x}_k = (0.5)^k \mathbf{v}_2 \to \mathbf{0}$.
In this example we see different behaviour in different directions. We describe this by saying that the origin is a *saddle point*.

Here are some trajectories with different starting points:



saddle

If a starting point is closer to $\mathbf{v}_2$ it is initially attracted to the origin, and when it gets closer to $\mathbf{v}_1$ it is repelled. If the initial point is closer to $\mathbf{v}_1$, it

Dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} 1.25 & -.75 \\ -.75 & 1.25 \end{bmatrix}$$

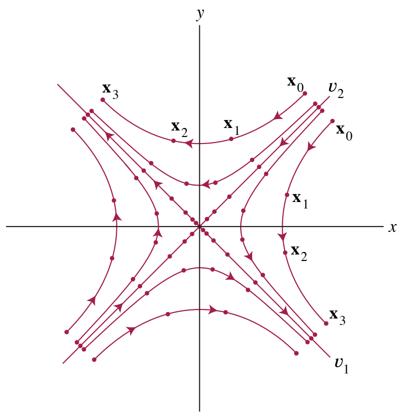


FIGURE 4 The origin as a saddle point.

Example 3

Describe the trajectories of the dynamical system associated to the matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

The characteristic polynomial for A is $(4-\lambda)^2 - 1 = \lambda^2 - 8\lambda + 15 = (\lambda - 5)(\lambda - 3)$. Thus the eigenvalues are 5 and 3 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Hence for any initial vector

$$\mathbf{x}_0 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

we have

$$\mathbf{x}_k = c_1 5^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 3^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

As k becomes large, so do both 5^k and 3^k. Hence $\mathbf{x}_k$ tends away from the origin.

Because the dominant eigenvalue 5 has corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, all trajectories for which $c_1 \neq 0$ will end up in the first or third quadrant.
Trajectories for which $c_2 = 0$ start and stay on the line $y = x$ whose direction vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (They move away from $\mathbf{0}$ along this line, unless $\mathbf{x}_0 = \mathbf{0}$).

Similarly, trajectories for which $c_1 = 0$ start and stay on the line $y = -x$ whose direction vector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

In this case $\mathbf{0}$ is called a repellor. This occurs whenever all eigenvalues have modulus greater than 1.

Dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$$

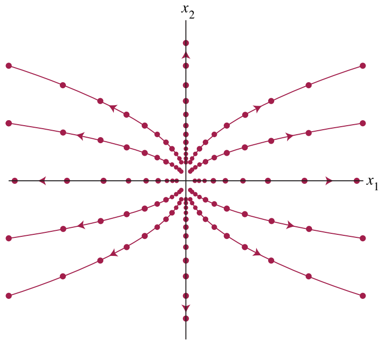


FIGURE 2 The origin as a repellor.

Example 4

Describe the trajectories of the dynamical system associated to the matrix $A = \begin{bmatrix} 0.5 & 0.4 \\ -0.125 & 1.1 \end{bmatrix}$. (This was the final matrix in the owl/rat examples earlier.)

Here the eigenvalues 1 and 0.6 have associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. So we have

$$\mathbf{x}_k = c_1\mathbf{v}_1 + 0.6^k c_2\mathbf{v}_2 \,.$$

As $k \to \infty$, we have $\mathbf{x}_k$ approaching the fixed point $c_1\mathbf{v}_1$.
This situation is unstable – a small change to the entries can have a major effect on the behaviour.

For example with $A := \begin{bmatrix} 0.5 & 0.4 \\ (-0.125) & 1.1 \end{bmatrix}$

value	eigenvalue	eigenvalue	behaviour
-0.125	1	0.6	$\mathbf{x}_k \to c_1\mathbf{v}_1$
-0.1249	1.0099	0.5990	*saddle point*
-0.1251	0.9899	0.6010	$\mathbf{x}_k \to 0$

This example comes from a model of populations of a species of owl and its prey (Lay 5.6.4). In spite of the model being very simplistic, the ecological implications of instability are clear.

Complex eigenvalues

What about trajectories in the complex situation?
Consider the matrices

$$(a) \quad A = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad \text{eigenvalues } \lambda = \tfrac{1}{2} + i\tfrac{1}{2},\ \overline{\lambda} = \tfrac{1}{2} - i\tfrac{1}{2}$$

where $|\lambda| = |\overline{\lambda}| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} < 1.$

$$(b) \quad A = \begin{bmatrix} 0.2 & -1.2 \\ 0.6 & 1.4 \end{bmatrix}, \quad \text{eigenvalues } \lambda = \tfrac{4}{5} + i\tfrac{3}{5},\ \overline{\lambda} = \tfrac{4}{5} - i\tfrac{3}{5}$$

where $|\lambda| = |\overline{\lambda}| = \sqrt{(\frac{4}{5})^2 + (\frac{3}{5})^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{1} = 1.$

If we plot the trajectories beginning with $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ for the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, we get some interesting results.

In case (a) the trajectory spirals into the origin, whereas for (b) it appears to follow an elliptical orbit.

For matrices with complex eigenvalues we can summarise as follows:
if A is a real 2×2 matrix with complex eigenvalues $\lambda = a \pm bi$ then the trajectories of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$

- spiral inward if $|\lambda| < 1$ ($\mathbf{0}$ is a spiral attractor),
- spiral outward if $|\lambda| > 1$ ($\mathbf{0}$ is a spiral repellor),
- and lie on a closed orbit if $|\lambda| = 1$ ($\mathbf{0}$ is a orbital centre).

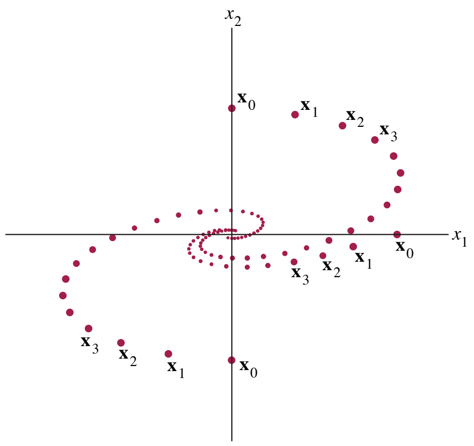


FIGURE 5 Rotation associated with complex eigenvalues.

Some further examples

Example 5

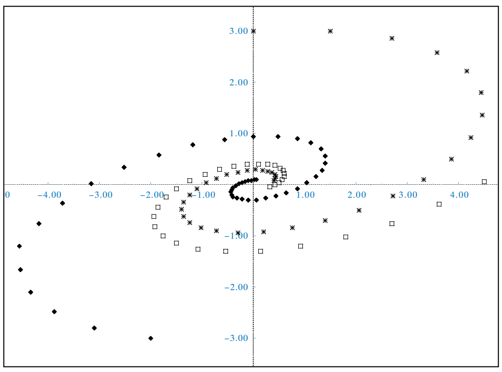
Let $A = \begin{bmatrix} 0.8 & 0.5 \\ -0.1 & 1.0 \end{bmatrix}$.

Here the eigenvalues are $0.9 \pm 0.2i$, with eigenvectors $\begin{bmatrix} 1 \mp 2i \\ 1 \end{bmatrix}$. As we noted in Section 18, setting $P = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, $\cos\varphi = \frac{0.9}{\sqrt{0.85}}$, $\sin\varphi = \frac{0.2}{\sqrt{0.85}}$,

$$P^{-1}AP = \begin{bmatrix} 0.9 & -0.2 \\ 0.2 & 0.9 \end{bmatrix} = \sqrt{0.85}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

a scaling (approximately 0.92) and a rotation (through approximately 44°).

$P^{-1}AP$ is the matrix of T_A with respect to the basis of the columns of P.
Note that the rotation is anticlockwise.
Here are the trajectories with respect to the original axes. They go clockwise, indicated by $\det(P) < 0$.



Example 6

(Lay 5.6.18) In a herd of buffalo, there are adults, yearlings and calves. On average 42 female calves are borne to every 100 adult females each year, 60% of the female calves survive to become yearlings, and 75% of the female yearlings survive to become adults, and 95% of the adults survive to the next year.

This information gives the following relation:

$$\begin{bmatrix} \mathit{adults} \\ \mathit{year..s} \\ \mathit{calves} \end{bmatrix}_{k+1} = \begin{bmatrix} 0.95 & 0.75 & 0 \\ 0 & 0 & 0.60 \\ 0.42 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathit{adults} \\ \mathit{year..s} \\ \mathit{calves} \end{bmatrix}_{k}$$

Assuming that there are sufficient adult males, what are the long term prospects for the herd?

Eigenvalues are approximately

$$1.1048, -0.0774 \pm 0.4063i.$$

The complex eigenvalues have modulus approximately 0.4136.

Corresponding eigenvectors are approximately $\mathbf{v}_1 = \begin{bmatrix} 100.0 \\ 20.65 \\ 38.0 \end{bmatrix}$, and a complex conjugate pair $\mathbf{v}_2, \mathbf{v}_3$.

Thus in the complex setting

$$\begin{aligned} \mathbf{x}_k = \quad 1.1048^k c_1 \mathbf{v}_1 \quad &+ (-0.0774 + 0.4063i)^k c_2 \mathbf{v}_2 \\ &+ (-0.0774 - 0.4063i)^k c_3 \mathbf{v}_3. \end{aligned}$$

The last two terms go to $\mathbf{0}$ as $k \to \infty$, so in the long term the population of females is determined by the first term, which grows at about 10.5% a year. The distribution of females is 100 adults to 21 yearlings to 38 calves. □

Survival of the Spotted Owls

In the introduction to this chapter the survival of the spotted owl population is modelled by the system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ where

$$\mathbf{x}_k = \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix}$$

where $\mathbf{x}_k$ lists the numbers of females at time k in the juvenile, subadult and adult life stages.
Computations give that the eigenvalues of A are approximately $\lambda_1 = 0.98, \lambda_2 = -0.02 + 0.21i$, and $\lambda_3 = -0.02 - 0.21i$. All eigenvalues are less than 1 in magnitude, since $|\lambda_2|^2 = |\lambda_3|^2 = (-0.02)^2 + (0.21)^2 = 0.0445$.

Denote corresponding eigenvectors by $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. the general solution of $\mathbf{x}_{k+1} = A\mathbf{x}_k$ has the form

$$\mathbf{x}_k = c_1(\lambda_1)^k\mathbf{v}_1 + c_2(\lambda_2)^k\mathbf{v}_2 + c_3(\lambda_3)^k\mathbf{v}_3.$$

Since all three eigenvalues have magnitude less than 1, all the terms on the right of this equation approach the zero vector. So the sequence $\mathbf{x}_k$ also approaches the zero vector.
So this model predicts that the spotted owls will eventually perish.

However if the matrix describing the system looked like

$$\begin{bmatrix} 0 & 0 & 0.33 \\ 0.3 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \quad \text{instead of} \quad \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix}$$

then the model would predict a slow growth in the owl population. The real eigenvalue in this case is $\lambda_1 = 1.01$, with $|\lambda_1| > 1$.
The higher survival rate of the juvenile owls may happen in different areas from the one in which the original model was observed.