Overview

Last time we discussed orthogonal projection. We'll review this today before discussing the question of how to find an orthonormal basis for a given subspace.

From Lay, §6.4

Orthogonal projection

Given a subspace W of \mathbb{R}^n , you can write any vector $\mathbf{y} \in \mathbb{R}^n$ as

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = \operatorname{proj}_W \mathbf{y} + \operatorname{proj}_{W^{\perp}} \mathbf{y},$$

Second Semester 2015

Second Semester 2015 2 / 24

Second Semester 2015

3 / 24

1 / 24

where $\hat{\mathbf{y}} \in W$ is the closest vector in W to \mathbf{y} and $\mathbf{z} \in W^{\perp}$. We call $\hat{\mathbf{y}}$ the orthogonal projection of \mathbf{y} onto W.

Given an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ for W, we have a formula to compute $\hat{\mathbf{y}}$:

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$$

If we also had an orthogonal basis $\{\mathbf{u}_{p+1},\ldots,\mathbf{u}_n\}$ for \mathcal{W}^{\perp} , we could find \mathbf{z} by projecting **y** onto W^{\perp} :

$$\mathsf{z} = \frac{\mathsf{y} \cdot \mathsf{u}_{p+1}}{\mathsf{u}_{p+1} \cdot \mathsf{u}_{p+1}} \mathsf{u}_{p+1} + \dots + \frac{\mathsf{y} \cdot \mathsf{u}_n}{\mathsf{u}_n \cdot \mathsf{u}_n} \mathsf{u}_n.$$

However, once we subtract off the projection of \mathbf{y} to W, we're left with $z \in W^{\perp}$. We'll make heavy use of this observation today. Dr Scott Morrison (ANU) MATH1014 Not

Orthonormal bases

In the case where we have an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ for W, the computations are made even simpler:

$$\hat{\mathbf{y}} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_{\rho})\mathbf{u}_{\rho}.$$

If $\mathcal{U} = \{\mathbf{u_1}, \dots, \mathbf{u_p}\}$ is an orthonormal basis for W and U is the matrix whose columns are the u_i , then

- $UU^T \mathbf{y} = \hat{\mathbf{y}}$
- $U^T U = I_p$

The Gram Schmidt Process

The aim of this section is to find an orthogonal basis $\{v_1,\ldots,v_n\}$ for a subspace W when we start with a basis $\{x_1,\ldots,x_n\}$ that is not orthogonal.

Start with $\mathbf{v}_1 = \mathbf{x}_1$.

Now consider $x_2.$ If v_1 and x_2 are not orthogonal, we'll modify x_2 so that we get an orthogonal pair $v_1,\,v_2$ satisfying

 $\mathsf{Span}\{x_1, x_2\} = \mathsf{Span}\{v_1, v_2\}.$

Then we modify x_3 so get v_3 satisfying $v_1 \cdot v_3 = v_2 \cdot v_3 = 0$ and

 $\mathsf{Span}\{x_1, x_2, x_3\} = \mathsf{Span}\{v_1, v_2, v_3\}.$

We continue this process until we've built a new orthogonal basis for W.

| Dr Scott Morrison (ANU) | MATH1014 Notes | Second Semester 2015 4 / 24 |
|---|---|--|
| | | |
| | | |
| Example 1 | | |
| Suppose that $W = \text{Span}$ orthogonal basis { $\mathbf{v}_1, \mathbf{v}_2$ } | $\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 1\\ 1\\ 0 \end{bmatrix}$ for W . | and $\mathbf{x}_2 = \begin{bmatrix} 2\\ 2\\ 3 \end{bmatrix}$. Find an |
| To start the process we p We then find | put $\mathbf{v}_1 = \mathbf{x}_1$. | |
| $\hat{\mathbf{y}} = proj$ | $\mathbf{v}_1 \mathbf{x}_2 = \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \frac{4}{2} \begin{bmatrix} 1\\1\\0 \end{bmatrix} \mathbf{v}_1$ | $= \begin{bmatrix} 2\\2\\0\end{bmatrix}.$ |
| | | |
| Dr Scott Morrison (ANU) | MATH1014 Notes | Second Semester 2015 5 / 24 |
| | | |

Now we define $\mathbf{v}_2 = \mathbf{x}_2 - \hat{\mathbf{y}}$; this is orthogonal to $\mathbf{x}_1 = \mathbf{v}_1$:

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \mathbf{x}_2 - \hat{\mathbf{y}} = \begin{bmatrix} 2\\2\\3 \end{bmatrix} - \begin{bmatrix} 2\\2\\0 \end{bmatrix} = \begin{bmatrix} 0\\0\\3 \end{bmatrix}.$$

So \mathbf{v}_2 is the component of \mathbf{x}_2 orthogonal to \mathbf{x}_1 . Note that \mathbf{v}_2 is in $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ because it is a linear combination of $\mathbf{v}_1 = \mathbf{x}_1$ and \mathbf{x}_2 . So we have that

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 1\\1\\0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0\\0\\3 \end{bmatrix} \right\}$$

Second Semester 2015

6 / 24

is an orthogonal basis for W.

Dr Scott Morrison (ANU)

Example 2

Suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for a subspace W of \mathbb{R}^4 . Describe an orthogonal basis for W.

• As in the previous example, we put

$$\mathbf{v}_1 = \mathbf{x}_1 \quad \text{and} \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$$

Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for $W_2 = \text{Span} \{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span} \{\mathbf{v}_1, \mathbf{v}_2\}$.

• Now $\operatorname{proj}_{W_2} \mathbf{x}_3 = \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2$ and

$$\mathbf{v}_3 = \mathbf{x}_3 - \operatorname{proj}_{W_2} \mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2$$

Second Semester 2015 7 / 24

Second Semester 2015

Second Semester 2015 9 / 24

8 / 24

is the component of \mathbf{x}_3 orthogonal to W_2 . Furthermore, \mathbf{v}_3 is in W because it is a linear combination of vectors in W.

• Thus we obtain that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W.

Theorem (The Gram Schmidt Process)

Given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_p\}$ for a subspace W of \mathbb{R}^n , define

$$\mathbf{v}_{1} = \mathbf{x}_{1}$$

$$\mathbf{v}_{2} = \mathbf{x}_{2} - \frac{\mathbf{x}_{2} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1}$$

$$\mathbf{v}_{3} = \mathbf{x}_{3} - \frac{\mathbf{x}_{3} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} - \frac{\mathbf{x}_{3} \cdot \mathbf{v}_{2}}{\mathbf{v}_{2} \cdot \mathbf{v}_{2}} \mathbf{v}_{2}$$

$$\vdots$$

$$\mathbf{v}_{p} = \mathbf{x}_{p} - \frac{\mathbf{x}_{p} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} - \dots - \frac{\mathbf{x}_{p} \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}} \mathbf{v}_{p-1}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W. Also

Span
$$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$$
 = Span $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for $1 \le k \le p$.

Example 3

Dr Scott Morrison (ANU)

The vectors

$$\mathbf{x}_1 = \begin{bmatrix} 3\\ -4\\ 5 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -3\\ 14\\ -7 \end{bmatrix}$$

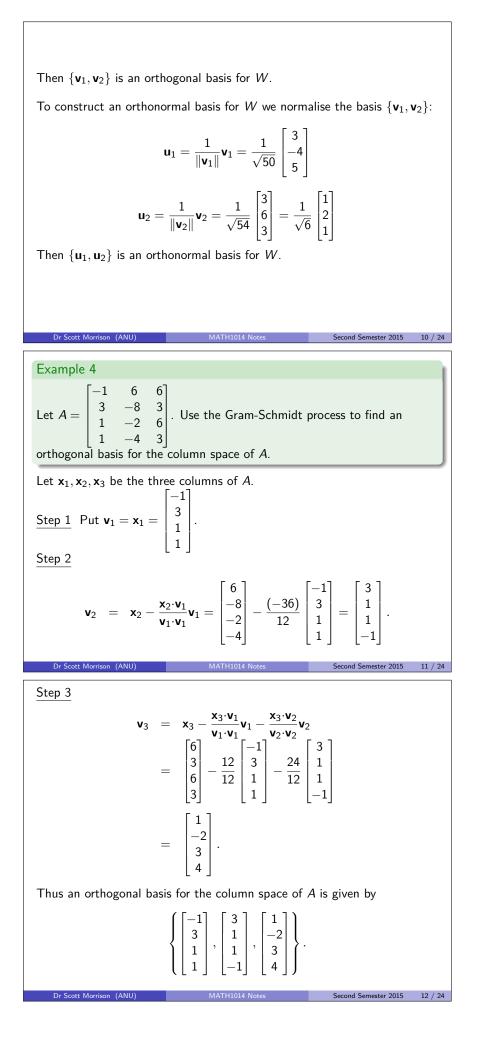
form a basis for a subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W.

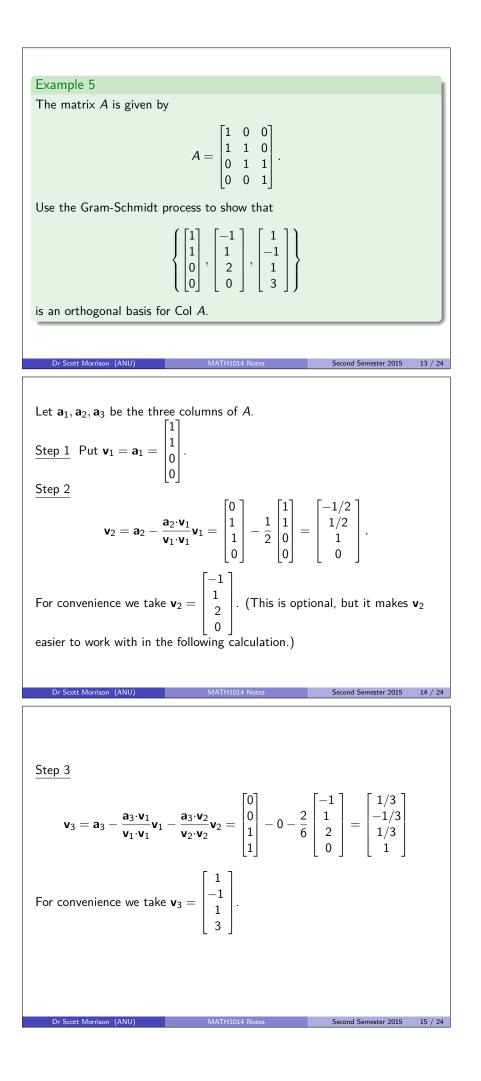
 $\frac{\text{Step 1}}{\text{Step 2}} \ \ \, \text{Put } \mathbf{v}_1 = \mathbf{x}_1.$

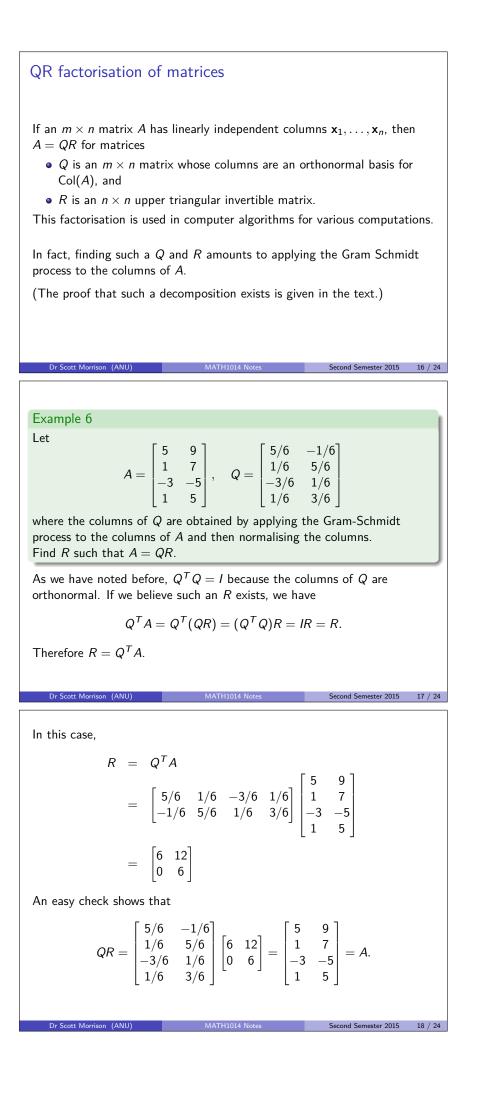
$$\mathbf{v}_{2} = \mathbf{x}_{2} - \frac{\mathbf{x}_{2} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1}$$
$$= \begin{bmatrix} -3\\ 14\\ -7 \end{bmatrix} - \frac{(-100)}{50} \begin{bmatrix} 3\\ -4\\ 5 \end{bmatrix} = \begin{bmatrix} 3\\ 6\\ 3 \end{bmatrix}.$$

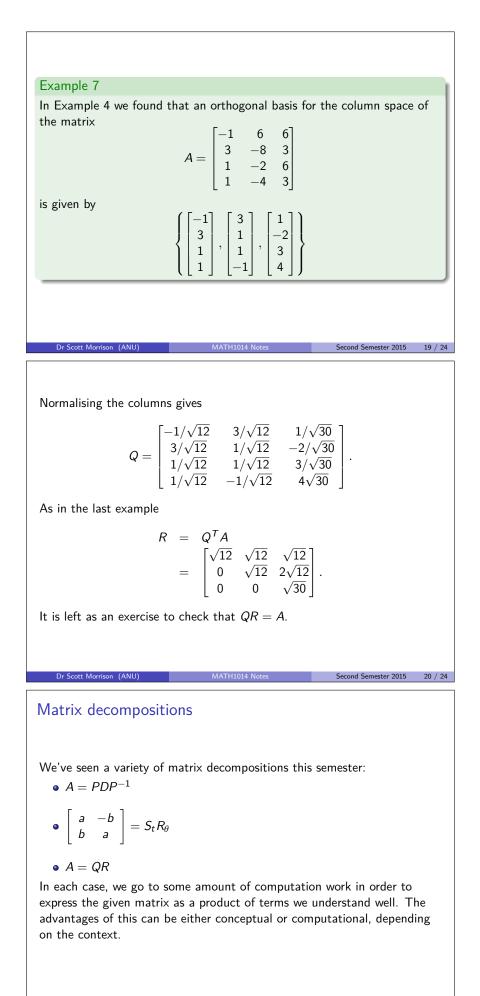
MATH1014 Notes

Dr Scott Morrison (ANU)









Second Semester 2015 21 / 24

Dr Scott Morrison (ANU)

