Some Revision Questions

1. Consider the following two bases for $\mathbb{R}^2$
$$\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \right\}$$
Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

2. Find the eigenvalues and eigenvectors for the matrix
$$A = \begin{bmatrix} 3 & -2 & 8 \\ 0 & 5 & -2 \\ 0 & -4 & 3 \end{bmatrix}.$$
Determine if A is diagonalisable, and if so find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

3. Find all the real values of k for which the matrix A is diagonalisable.
$$(i)\ A = \begin{bmatrix} 1 & k & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (ii)\ A = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Show that the matrices A and B are similar by showing that they are similar to the same diagonal matrix. Then find an invertible matrix P such that $P^{-1}AP = B$.
$$A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

5. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

   (a) Sketch the first six points of the trajectory for the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ taking $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. From this would you classify the the origin as a spiral attractor, spiral repellor, or orbital centre?

   (b) Find an invertible matrix P and a matrix C of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$.

6. Let $A = \begin{bmatrix} -3 & 2 \\ -1 & -5 \end{bmatrix}$. Find the (complex) eigenvalues and a basis for each eigenspace.

7. Find the orthogonal projection of $\mathbf{v}$ onto the subspace W of $\mathbb{R}^4$ spanned by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.
$$\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 4 \\ -3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$
Find the distance from $\mathbf{v}$ to W.

8. Find all possible values of a, b in $\mathbb{R}$ for which the 2×2 matrix

$$U = \begin{bmatrix} a & \frac{2}{\sqrt{5}} \\ b & \frac{1}{\sqrt{5}} \end{bmatrix}$$

is orthogonal.

9. A dynamical system is described by the matrix equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ where the matrix A is given by

$$A = \begin{bmatrix} 0.5 & 0.2 \\ -0.5 & 1.2 \end{bmatrix}.$$

The matrix A has eigenvalues 1 and 0.7.

(a) Find the eigenvectors of A.

(b) If $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$, find the long term behaviour of the dynamical system.

10. On any given day, a student is either healthy or ill. Of the students who are healthy today, 90% will be healthy tomorrow. Of the students who are ill today, 30% will be ill tomorrow.

(a) Construct the stochastic matrix for this situation.

(b) Suppose that 20% of the students are ill on Monday. What percentage of the students are likely to be ill on Wednesday?

(c) In the long run what fraction of the students are expected to be healthy?

11. $T : M_{2\times 2} \to M_{2\times 2}$ is given by

$$T(A) = AB - BA$$

where $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Find the matrix of T with respect to the“standard” basis for $M_{2\times 2}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(b) Find a basis for the kernel of T.

(c) Explain why T is not one to one.

(d) Find a basis for the range of T.

(e) Explain why T is not onto.

12. $T : \mathbb{P}_2 \to \mathbb{P}_2$ is given by

$$T(p(x)) = p(3x + 2).$$

(a) Find the matrix of T with respect to the standard basis for $\mathbb{P}_2$.

(b) If possible find a basis for $\mathbb{P}_2$ for which the matrix of T is a diagonal matrix.

13. Consider the vector space W given by $W = \text{Span } \{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\}$.
Let $D : W \to W$ be the differential operator defined by $D(f(x)) = f'(x)$ for every $f(x) \in W$ (where $f'(x)$ is the derivative of $f(x)$).

   (a) Find the matrix of D with respect to $\mathcal{B} = \{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\}$.

   (b) Compute the derivative of $f(x) = 3e^{2x} - 3e^{2x}\cos x + 5e^{2x}\sin x$ using the matrix you have just constructed in part (a).

   (c) Use the matrix in part (a) to find $\int (2e^{2x}\cos x - 4e^{2x}\sin x)\, dx$.