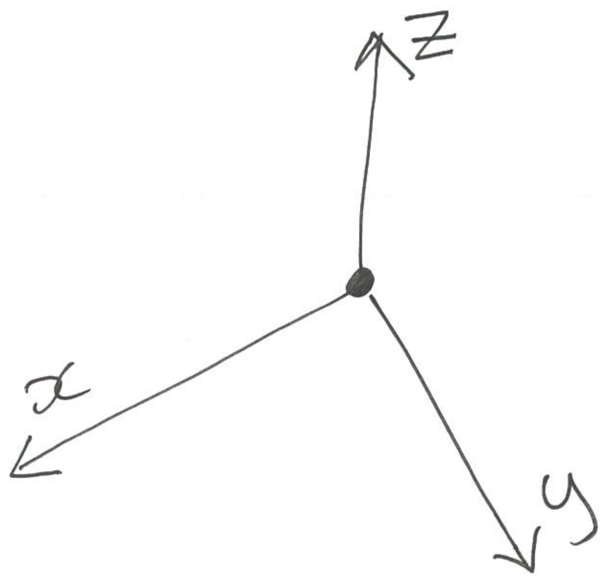
Euclidean space and coordinate systems

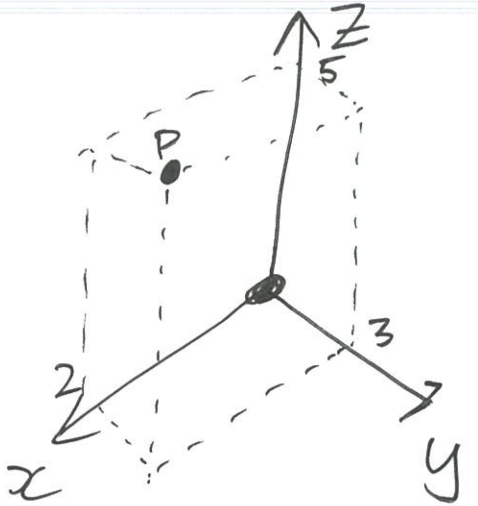
To describe 3-dimensional space, we begin by choosing a point called the origin, and then three directed lines through the origin which are perpendicular to each other, which we call the coordinate axes, or the x-axis, y-axis, and z-axis.

We usually think of the x-axis and y-axis as being horizontal (giving us the usual picture of 2-dimensional space), and the z-axis as being vertical.

We make sure to choose the directions of these lines so we have a right-handed coordinate system.



We can then represent any point in 3-dimesional space by giving its coordinates with respect to these axes.



$P = (2, 3, 5)$

In this way, we <u>identify</u> 3-dimensional space with $\mathbb{R}^3$, the set of ordered triples of real numbers.

$$\mathbb{R}^3 = \{ (x, y, z) \mid x, y, z \in \mathbb{R} \}$$

Distance formulas

In $\mathbb{R}^2$, we have the distance between two points

$$P_1 = (x_1, y_1) \quad \text{and} \quad P_2 = (x_2, y_2)$$

given by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

In $\mathbb{R}^3$, if $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, then

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Surfaces in $\mathbb{R}^3$

<u>Example</u> The sphere of radius r with centre $C=(c_1, c_2, c_3)$ is the set of all points in $\mathbb{R}^3$ with distance r from C:

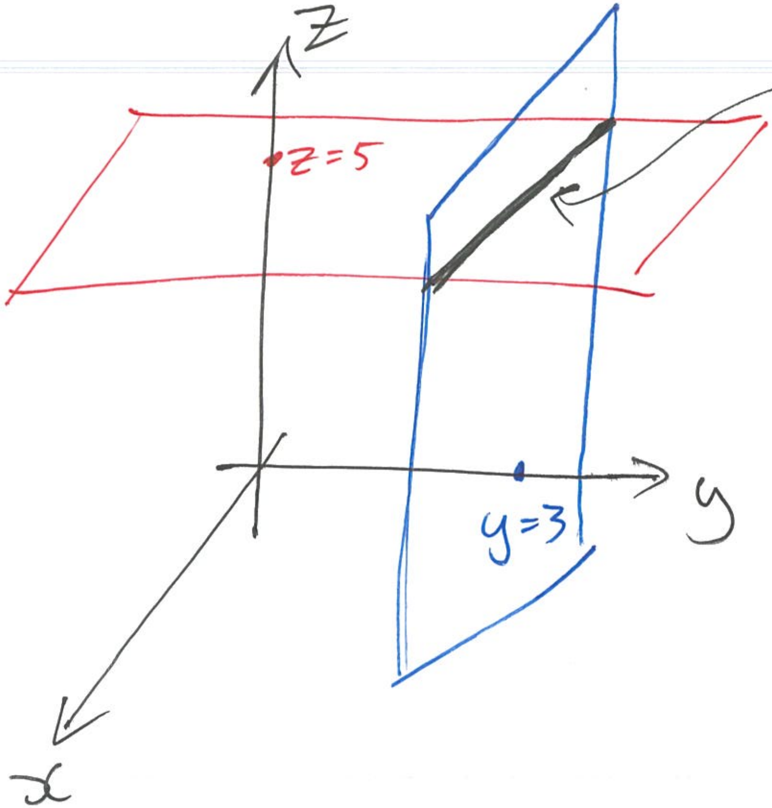
$$S = \{ P : |PC| = r \}$$

Equivalently, the sphere consists of all solutions to

$$(x-c_1)^2 + (y-c_2)^2 + (z-c_3)^2 = r^2.$$

Example

What does the pair of equations $y=3$, $z=5$ represent?



the line $\{(x,3,5) : x \in \mathbb{R}\}$.

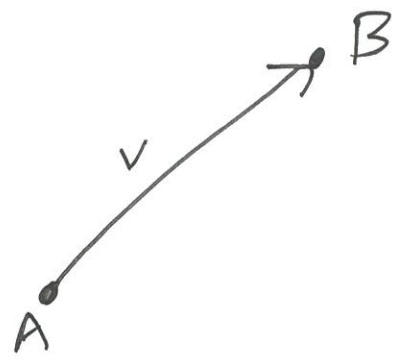
A single linear equation in $\mathbb{R}^3$ usually has solutions which form a plane.
A pair of — " — a line.
3 linear equations in $\mathbb{R}^3$ — " — a point.

When do these rules fail?

Vectors

A vector has magnitude and direction.

We represent vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ by arrows:



This is the vector v, which stretches from the point A to the point B.
We might also write it as $v = \overrightarrow{AB}$.

The location of a vector doesn't matter — we can slide it around (without stretching or rotating!) and it's still considered the same vector.

Vector addition

To add two vectors, we place them head to tail

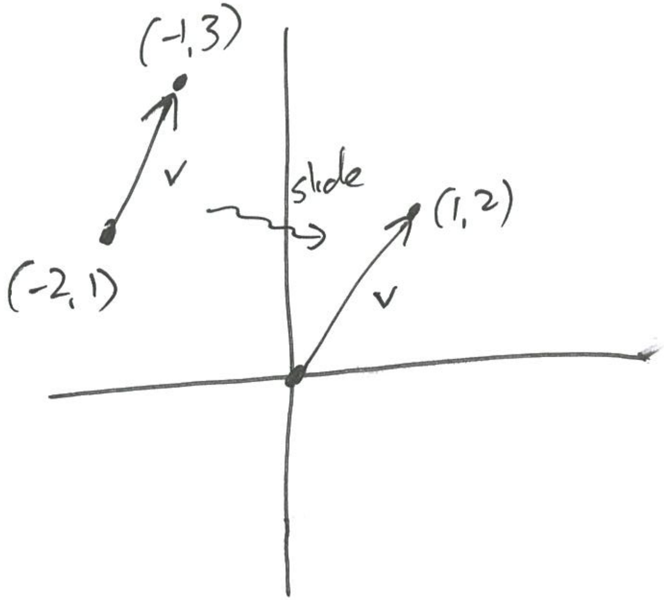
$v =$ ↗ , $w =$ ↓ $\Longrightarrow$ $v + w =$ 

To multiply a vector by a scalar, we change the magnitude of the vector by that ~~sa~~ factor, without changing the direction.

If $v =$ ↗, $3v =$ ↗, $-v =$ ↙.

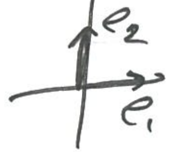
In coordinates, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$, and $k\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}$.

If we draw a vector with tail at the origin, we can use the <u>coordinates</u> of the head to describe the vector. These are called the **components** of the vector.



$$\Longrightarrow \quad v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The **standard basis vector** in $\mathbb{R}^2$ are $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



and in $\mathbb{R}^3$ are $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

If a vector $v \in \mathbb{R}^3$ has components $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we can write it in terms of the standard basis vectors as

$$v = ai + bj + ck.$$

The length of $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is $\sqrt{a^2+b^2+c^2}$, which we write as $\|v\|$.

If the length is 1, we call the vector a unit vector.

Finally, as long as v is not the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (which has zero length), then

$\dfrac{v}{\|v\|}$ is a unit vector pointing in the same direction as v.

Example

A river flows north at 1 km/hr, and a swimmer swims at 2 km/hr (relative to the water!)

- At what angle must the swimmer swim to move directly east across the river?
- How fast do they then move relative to the land?

Let F be the velocity of the river.

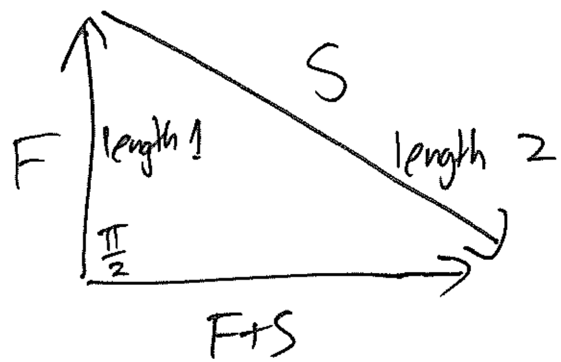
- It has magnitude 1, and points north.

Let S be the velocity of the swimmer.

- It has magnitude 2, but we don't know the direction.

We know $F+S$ points east.

We want to know
- the direction of S
- the magnitude of $F+S$.



Some geometry tells us the angle from F to S is $\frac{2\pi}{3}$, and the resulting speed is $\sqrt{3}$ km/hr.