Last time: coordinate systems and vectors in $\mathbb{R}^3$.
vectors can be added to each other,
and multiplied by scalars.

Does it make sense to "multiply" vectors?

Today we'll talk about the <u>dot product</u> in $\mathbb{R}^n$,
and the <u>cross product</u> in $\mathbb{R}^3$.

The <u>dot</u> or <u>scalar</u> product of vectors

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \text{is} \quad a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

<u>Example</u> $\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix} = (1)(-4) + (4)(5) + (-2)(-1) = -4 + 20 + 2 = 18$

Straight from the definition we have the following properties

1) $u \cdot v = v \cdot u$

2) $u \cdot (v + w) = u \cdot v + u \cdot w$

3) $k(u \cdot v) = (ku) \cdot v = u \cdot (kv)$ for $k \in \mathbb{R}$.

The dot product is related to lengths and angles:

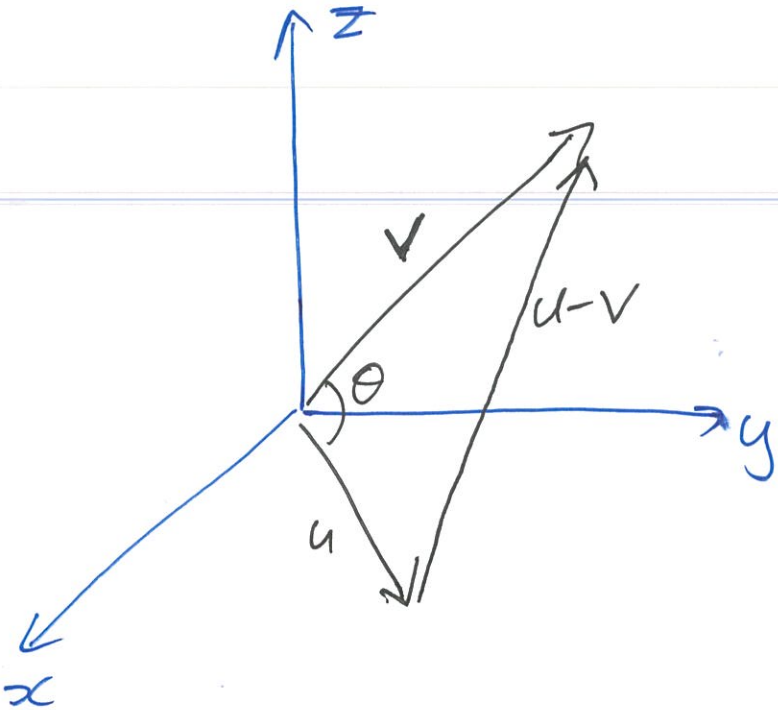
$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{v \cdot v}.$$

Theorem If θ is the angle between vectors $u, v \in \mathbb{R}^3$ (so $\theta \in [0, \pi]$), then

$$u \cdot v = \|u\| \|v\| \cos\theta$$

Definition Two vectors are orthogonal or perpendicular if their dot product is zero (i.e. $\theta = \frac{\pi}{2}$)

To see this, draw a diagram:



Now $\|u-v\|^2 = (u-v)\cdot(u-v)$

$= u\cdot u - u\cdot v - v\cdot u + v\cdot v$

$= u\cdot u + v\cdot v - 2u\cdot v$

$= \|u\|^2 + \|v\|^2 - 2u\cdot v.$

The cosine rule tells us

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$$

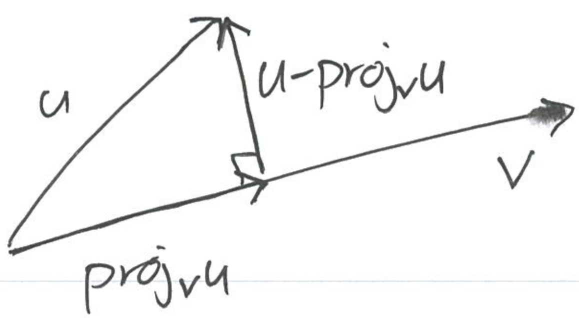
So $\quad u\cdot v = \|u\|\|v\|\cos\theta$

(Notice this still makes sense if u or v is zero, even though the angle then isn't well defined.)

Scalar and vector projections

The vector projection $\text{proj}_v u$ of a vector u in the direction of the nonzero vector v is the unique scalar multiple of v so that

$u - \text{proj}_v u$ is orthogonal to v.



It is the 'component of u in the v direction'.

It is given by the formula

$$\text{proj}_v u = \frac{u \cdot v}{\|v\|^2} v.$$

The scalar projection $\text{comp}_v u$ of u in the direction of v is just the length of the vector projection:

$$\text{comp}_v u = \frac{u \cdot v}{\|v\|} = \|u\| \cos\theta.$$

Cross product

In $\mathbb{R}^3$ only, there is another product called the cross product or vector product.

The cross product of $a, b \in \mathbb{R}^3$ is written $a \times b$, and is another vector in $\mathbb{R}^3$.

Definition

Given $a, b \in \mathbb{R}^3$, with angle $\theta \in [0, \pi]$ between them, the cross product is the unique vector $a \times b$ satisfying:

1) $|a \times b| = |a||b| \sin\theta$
2) $a \times b$ is orthogonal to both a and b
3) $\{a, b, a \times b\}$ forms a right handed coordinate syst (unless $\theta = 0$).

How can we find the components of $a \times b$ if

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}?$$

<u>Claim</u> $$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}.$$

(That is, this satisfies the 3 conditions!)

Recall how 2×2 determinants work: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

This generalises as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Notice $a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k.$

In light of this, we can write

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

even though strictly speaking this doesn't make sense.

<u>Example</u>

~~Find~~ Find a vector with positive k component perpendicular to both $a = 2i - j - 2k$ and $b = 2i - 3j + k$.

The cross product will be orthogonal to a

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = i \begin{vmatrix} -1 & -2 \\ -3 & 1 \end{vmatrix} - j \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} + k \begin{vmatrix} 2 & \\ 2 & - \end{vmatrix}$$

$$= -7i - 6j - 4k.$$

The negative of this, $\begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}$, will do!

Properties of the cross product

<u>Lemma</u> two nonzero vectors a and b are parallel (or antiparalle

iff $a \times b = 0$.

If $u, v, w \in \mathbb{R}^3$, and $t \in \mathbb{R}$, then

1. $u \times v = -v \times u$
2. $(u+v) \times w = u \times w + v \times w$
3. $u \times (v+w) = u \times v + u \times w$
4. $(tu) \times v = u \times (tv) = t(u \times v)$
5. $u \cdot (v \times w) = (u \times v) \cdot w$
6. $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$.

Notice its not associative: in general $(u \times v) \times w \neq u \times (v \times w)$.

Example

For which values of m do the points

$A = (1, 1, -1)$

$B = (0, 3, -2)$

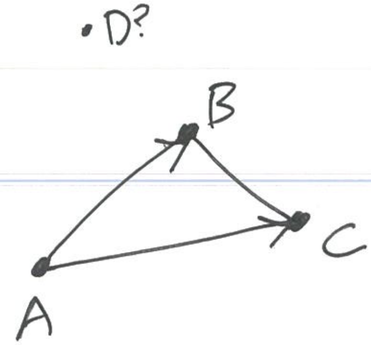
$C = (-2, 1, 0)$ and

$D = (m, 0, 2)$

all lie in a plane?

ABC form a triangle, which lies in some plan

We want D to lie in this plane as well.



If we can find a vector u orthogonal to

$\vec{AB}$ and to $\vec{AC}$,

then a point D lies in the plane spanned b

ABC if and only if

$\vec{AD}$ is also orthogonal to u.

We can find such a u as $u = \vec{AB} \times \vec{AC}$.

We want $\vec{AD} \cdot (\vec{AB} \times \vec{AC}) = 0$.

$$\vec{AB} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{AC} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \quad \vec{AD} = \begin{bmatrix} m-1 \\ -1 \\ 3 \end{bmatrix}.$$

Then $\vec{AD} \cdot (\vec{AB} \times \vec{AC}) = \begin{bmatrix} m-1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{vmatrix} i & j & k \\ -1 & 2 & -1 \\ -3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} m-1 & -1 & 3 \\ -1 & 2 & -1 \\ -3 & 0 & 1 \end{vmatrix}$

$$= -3\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} m-1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 15 + 2(m-1) - 1$$

$$= 2m + 12$$

So when $m=-6$, D lies in the plane spanned by ABC.