Lines and planes in $\mathbb{R}^3$

In $\mathbb{R}^2$, the general form for the equation of a line

is $$ax + by = c,$$

where a and b are not both zero.

If $b \neq 0$, we can write this as

$$y = \left(-\frac{a}{b}\right)x + \frac{c}{b}$$

which is of the form $y = mx + k$.

Here m is the slope of the line, and $(0,k)$ the y-intercept.

Example Let L be the line $x + 2y = 3$.

It has slope $-\frac{1}{2}$ and y-intercept $\left(0, \frac{3}{2}\right)$.

We could think of the line $y = -\frac{1}{2}x + \frac{3}{2}$ as the path of a moving particle.

At time $t=0$ it is at $(0, \frac{3}{2})$. Its x-coordinate increases by 1 unit per unit of time, while its y-coordinate decreases by $\frac{1}{2}$ unit per unit of time.

Thus at $t=1$ it is at $(1,1)$, while at $t=-2$ it was at $(-2, \frac{5}{2})$.

In general, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ \frac{3}{2} - \frac{1}{2}t \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} + t \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}.$$

What is the significance of this vector?

It is the <u>direction vector</u> for the line.

<u>Definition</u> The equation $r = r_0 + tv$ (here $r, r_0, v \in \mathbb{R}^n$, $t \in \mathbb{R}$) is the vector equation for a line. L

r_0 is the vector to any chosen point on L,
v is ~~the~~ a vector pointing in the direction of L.
t is the 'parameter'.

<u>Example</u> $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} + t \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$ is the vector equation of our line.

Write in coordinates, we have the parametric equations.

If $r = \begin{bmatrix} x \\ y \end{bmatrix}$, $r_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, $v = \begin{bmatrix} a \\ b \end{bmatrix}$,

$$r = r_0 + tv \iff \begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb. \end{aligned}$$

The vector equation of a line is ~~not~~ not unique.

We can change our 'base point' r_0 to any other point on the line,

or change our direction vector v by a nonzero scalar.

Example

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + t \begin{bmatrix} -4 \\ 8 \\ -10 \end{bmatrix}$$

all represent the same line.

(How do we check this? First check the direction vectors are parallel — or antiparallel — and the find the value of t showing that r_0 for the other line lies on the first line.

E.g. $$\begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}.$$

If all components of the direction vector are nonzero, we can eliminate t and obtain the symmetric equations.

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

$$\Updownarrow$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

<u>Example</u> Find the symmetric equations for the line thru $(1,2,3)$ in the direction $(2,3,-4)$.

The parametric form is $r = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$.

The symmetric equations are $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{y-3}{-4}$

Example

~~Determine~~

Describe the two lines

$$r = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

and

$$x + 1 = y - 4 = \frac{z-1}{3}.$$

If the lines intersect, there will be a value of t

So $$1 + 2t + 1 = 3t - 4 = \frac{2 - t - 1}{3}.$$

The first equation gives $t = 6$, but that doesn't satisfy the last equation.

So the lines don't intersect.

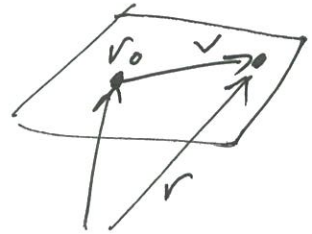
~~They~~ Their direction vectors are $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$,

so the lines aren't parallel either; they are skew lines.

Planes in $\mathbb{R}^3$.

We can describe a plane in a similar way.

If we pick a point r_0 on the plane, every other point r is of the form $r = r_0 + v$, for some vector v parallel to the plane.



If we find a vector n orthogonal to the plane, then v is parallel to the plane iff $n \cdot v = 0$.

Thus $n \cdot (r - r_0) = 0$ is the equation of the plane passing thru r_0, with normal vector n.

This is the vector equation of a plane.

Example

Find the vector equation of the plane passing thru $(0,-2,3)$, normal to $(4,2,-3)$

$n \cdot (r - r_0) = 0$ becomes

$$4(x-0) + 2(y--2) + 3(z-3) = 0$$

This is the scalar equation of the plane.

Once we expand everything out,

$$4x + 2y + 3z - 5 = 0$$

we get the standard form $(Ax + By + Cz + D = 0)$.

Notice that $D = 0$ iff the plane passes thru the origin.

<u>Example</u> Find the line thru the origin parallel to the line of intersection of the planes

$$x+2y-z=2$$

and

$$2x-y+4z=5.$$

First we extract the normal vectors

$$n_1 = (1, 2, -1)$$

and $n_2 = (2, -1, 4)$ for the two planes.

The line of intersection lies in both planes, so is orthogonal to both normal vectors.

Thus we can find its direction vector v as

$$v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & -1 & 4 \end{vmatrix} = i \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= 7i - 6j - 5k.$$

Thus the line has vector equation

$$r = t \begin{bmatrix} 7 \\ -6 \\ -5 \end{bmatrix},$$

parametric equations $x = 7t$, $y = -6t$, $z = -5t$

and symmetric equations $\frac{x}{7} = -\frac{y}{6} = -\frac{z}{5}$.