Yesterday: vector equations to describe lines and planes.

Line:

- $r = r_0 + tv$ — v: direction vector; r_0: some point on the line

Plane:

- $(r - r_0) \cdot n = 0$ — n: normal vector; r_0: some point on the plane

Today: how can we find the distance between:

- a point and a plane in $\mathbb{R}^3$
- a point and a line in $\mathbb{R}^3$
- two lines in $\mathbb{R}^3$, etc.

What does that even mean?

The distance between two points in $\mathbb{R}^3$ is the length of the line segment between them.

Definition The distance between any two subsets

$$A, B \subset \mathbb{R}^3$$

is the smallest distance between points a and b, such that $a \in A$ and $b \in B$.

To find the distance from a point P to a line L, we need to determine which point Q on the line is closest to P.

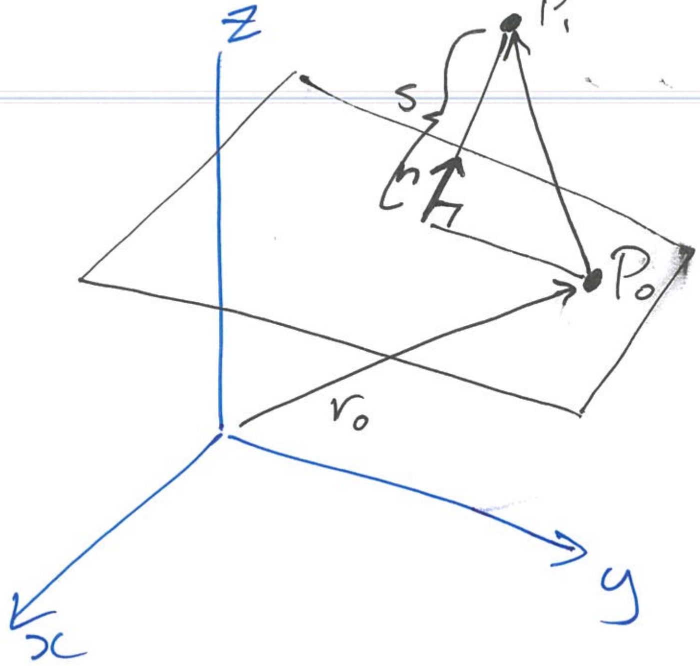
The segment PQ will be orthogonal to L.

The distance from P to L is the length of PQ.

Similarly for a plane.

Distance from a point to a plane.

What is the distance from a point $P_1=(x_1,y_1,z_1)$ and the plane $Ax+By+Cz+D=0$?



Consider $P_0=(x_0,y_0,z_0)$, any point on the plane.

The distance from P_1 to the plane is the (absolute value of the) scalar projection of $\overrightarrow{P_1P_0}$ onto the normal vector $n=\begin{bmatrix}A\\B\\C\end{bmatrix}$.

Thus $s = \left|\operatorname{comp}_n \overrightarrow{P_1P_0}\right|$

$$= \frac{\left|n \cdot \overrightarrow{P_1P_0}\right|}{\|n\|}$$

$$= \frac{\left|A(x_1-x_0)+B(y_1-y_0)+C(z_1-z_0)\right|}{\sqrt{A^2+B^2+C^2}}$$

$$= \frac{\left|Ax_1+By_1+Cz_1-(Ax_0+By_0+Cz_0)\right|}{\sqrt{A^2+B^2+C^2}}$$

Now (x_0, y_0, z_0) is on the plane, so $Ax_0+By_0+Cz_0+D=0$, and

$$s = \frac{\left|Ax_1+By_1+Cz_1+D\right|}{\sqrt{A^2+B^2+C^2}}.$$

Example

Find the distance from $(1,2,0)$ to the plane

$3x-4y-5z-2=0$

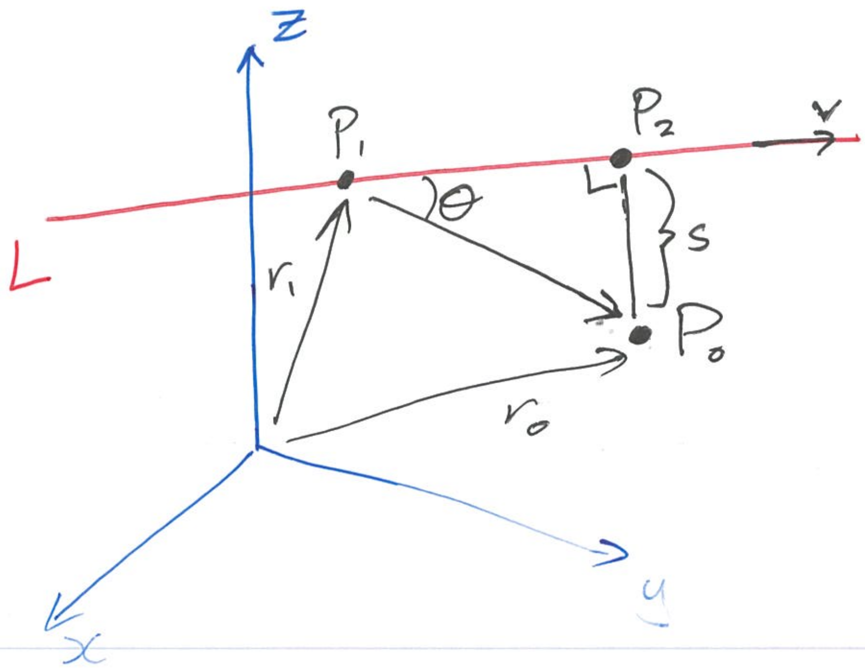
$$s=\frac{|3\cdot 1+-4\cdot 2+-5\cdot 0-2|}{\sqrt{3^2+4^2+5^2}}$$

$$=\frac{7}{5\sqrt{2}}.$$

Distance from a point to a line.

What is the distance s from a point $P_0 = (x_0, y_0, z_0)$ to the line L through $P_1 = (x_1, y_1, z_1)$ with direction vector v?

Write $r_0 = \overrightarrow{OP_0}$ and $r_1 = \overrightarrow{OP_1}$.



A point P_2 on L is closest to P_0 iff $\overrightarrow{P_2P_0}$ is orthogonal to L.

Call θ the angle between L and $\overrightarrow{P_1P_0}$.

The distance is given by

$$s = \|\overrightarrow{P_2P_0}\| = \|\overrightarrow{P_1P_0}\| \sin\theta = \|r_0 - r_1\| \sin\theta.$$

Now $\|(r_0 - r_1) \times v\| = \|r_0 - r_1\| \, \|v\| \sin\theta$, so

$$s = \frac{\|(r_0 - r_1) \times v\|}{\|v\|}.$$

Example

Find the distance from $(1,1,-1)$ to the line of intersection of the planes

$x+y+z=1$

$2x-y-5z=1$

The direction vector v is given by

$v = n_1 \times n_2$, with $n_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $n_2 = \begin{bmatrix}2\\-1\\-5\end{bmatrix}$.

$$v = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & -1 & -5 \end{vmatrix} = i\begin{vmatrix}1 & 1\\-1 & -5\end{vmatrix} - j\begin{vmatrix}1 & 1\\2 & -5\end{vmatrix} + k\begin{vmatrix}1 & 1\\2 & -1\end{vmatrix}$$

$$= \begin{bmatrix}-4\\7\\-3\end{bmatrix}.$$

Now we need r_1, ~~and~~ a point on the intersection.

$x = 1-y-z$

$2-2y-2z-y-5z=1$

$\cancel{2}+3y+7z=+1$

$y = \frac{1-7z}{3}$

$z=1,\ y=-2,\ x=2$

$r_0 = \begin{bmatrix}2\\-2\\1\end{bmatrix}.$

Distance between two lines

Suppose L_1 passes through P_1, with direction vector v_1,

and L_2 passes through P_2, with direction vector v_2.

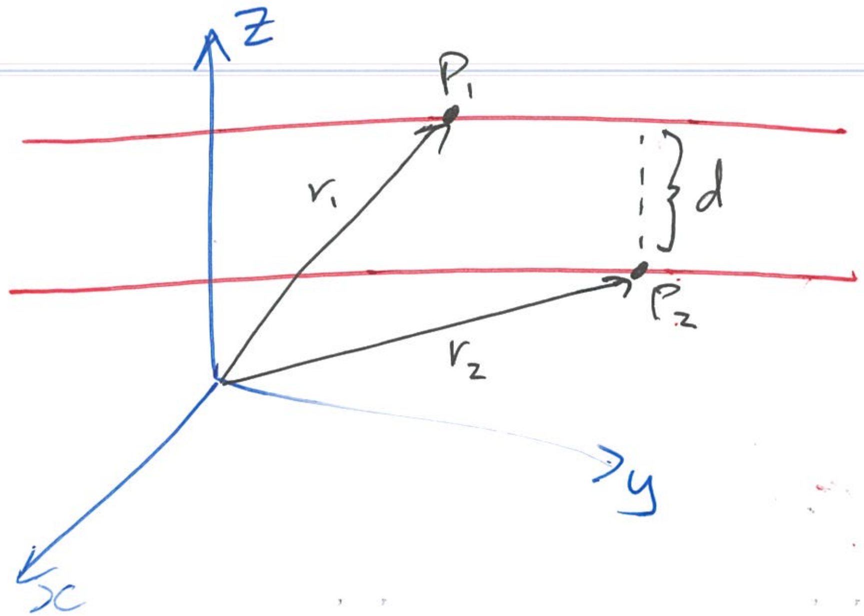
We want to compute the distance d between theses lines.

Either ① the lines intersect, and $d=0$

② the lines are parallel

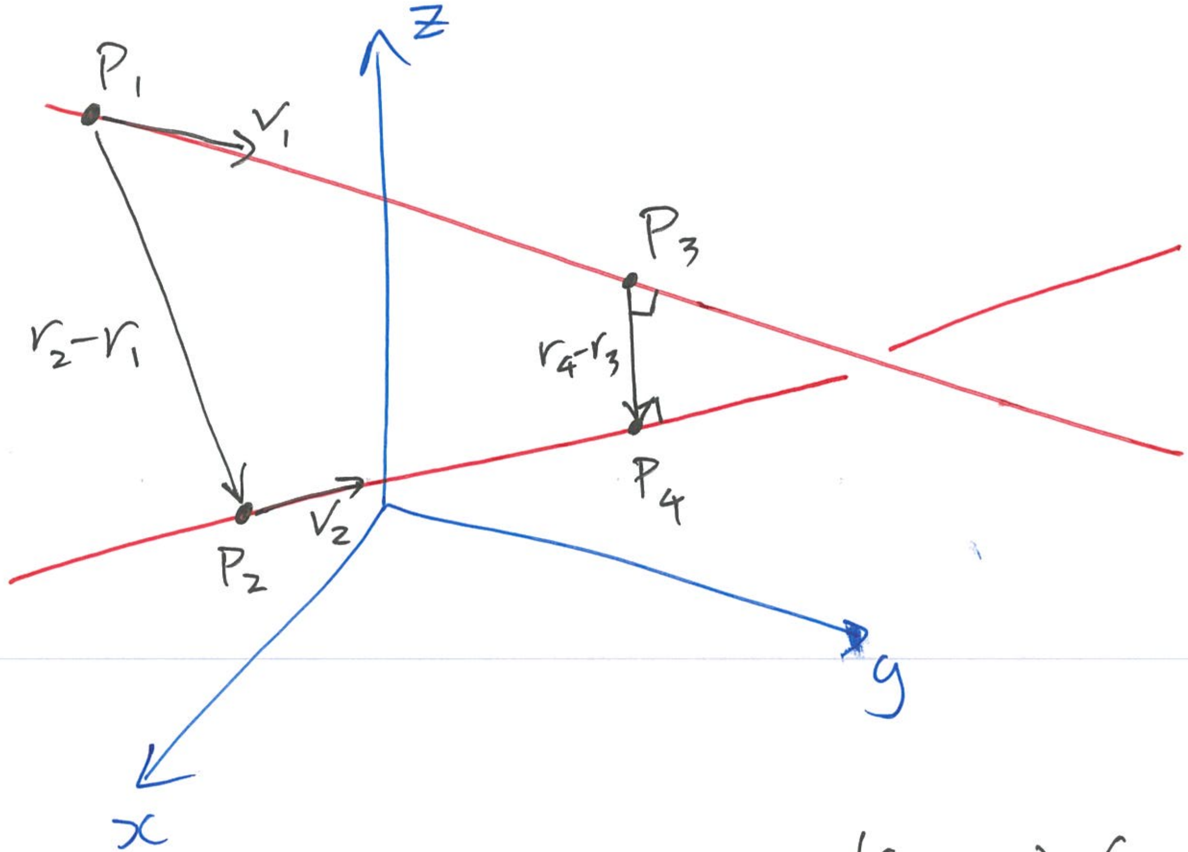
③ the lines are skew.

When the lines are parallel, the distance d is just given by the distance from P_2 to L_1



$$d = \frac{\|\overrightarrow{P_1P_2} \times v_1\|}{\|v_1\|} = \frac{\|(r_2 - r_1) \times v_1\|}{\|v_1\|}$$

When the lines are <u>skew</u>, we want to find P_3 on L_1 and P_4 on L_2, so P_3 and P_4 are as close as possible, then measure the distance $\|\overrightarrow{P_3P_4}\|$.



$d=\|r_4-r_3\|$ which we can rewrite as $d=\dfrac{|(r_4-r_3)\cdot(v_1\times v_2)|}{\|v_1\times v_2\|}$

since r_4-r_3 is orthogonal to both v_1 and v_2 (so parallel to $v_1\times v_2$).

What the point of that formula? We don't know r_3 or r_4!

Notice

$$r_4 = r_2 + t v_2 \quad \text{and} \quad r_3 = r_1 + s v_1$$

for some $s, t \in \mathbb{R}$.

Substituting, we obtain

$$d = \frac{|(r_2 - r_1 + t v_2 - s v_1) \cdot (v_1 \times v_2)|}{\|v_1 \times v_2\|}$$

$$= \frac{|(r_2 - r_1) \cdot (v_1 \times v_2)|}{\|v_1 \times v_2\|}$$

since v_2 and v_1 are both perpendicular to $v_1 \times v_2$.

(We still don't know r_4 and r_3! Can you find formulas for them?)

Example

Find the distance between the lines

$$L_1: \begin{cases} x+2y=3 \\ y+2z=3 \end{cases}$$

$$L_2: \begin{cases} x+y+z=6 \\ x-2z=-5 \end{cases}$$

$$V_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 4i - 2j + k$$

$$V_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{vmatrix} = -2i + 3j - k$$