"Abstract" vector spaces

- Many 'linear' systems we want to study do not live in $\mathbb{R}^3$, or indeed any $\mathbb{R}^n$!

- Abstract vector spaces will seem pretty 'abstract'!
  But it's precisely the level of generality we need to capture all the examples, and reason about them uniformly.

Definition

A vector space is a non-empty set V, whose elements are called <u>vectors</u>, along with operations

$$+ : V \times V \longrightarrow V \qquad \text{"vector addition"}$$

$$\cdot : \mathbb{R} \times V \longrightarrow V \qquad \text{"scalar multiplication"}$$

later in the course we'll also think about 'complex vector spaces'.

satisfying the following axioms:

- $u + v = v + u$ for all $u, v \in V$ (commutativity)
- $(u+v)+w = u+(v+w)$ for all $u, v, w \in V$ (associativity)
- there is a vector called '0', such that $0+u=u$ for all u.
- for each vector u, there is a vector called '$-u$' such that

$$u+(-u)=0$$

- $c(u+v) = cu + cv$ for all $c \in \mathbb{R}$, $u, v \in V$
- $(c+d)u = cu + du$ for all $c, d \in \mathbb{R}$, $u \in V$
- $c(du) = (cd)u$ for all $c, d \in \mathbb{R}$, $u \in V$
- $1u = u$ for all $u \in V$.

Examples

Let $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$, with the usual operations of matrix addition and multiplication by a scalar.

Here the zero vector $\mathbf{0}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The negative of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

$$t \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ta & tb \\ tc & td \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a+x & b+y \\ c+z & d+w \end{bmatrix}.$$

Let $\mathbb{P}_2$ denote the set of all polynomials with degree at most 2, with coefficients in $\mathbb{R}$.

$$\mathbb{P}_2 = \{ p(t) = a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R} \}.$$

We can add polynomials, or multiply by a scalar.

Non-example

The integers $\mathbb{Z}$ with the usual operations is not a vector space, because scalar multiplication can 'fall outside' of $\mathbb{Z}$:

$$\underset{\mathbb{R}}{\underset{\uparrow}{\left(\tfrac{1}{4}\right)}} \cdot \underset{\mathbb{Z}}{\underset{\uparrow}{(3)}} = \underset{\mathbb{Z}.}{\underset{\not\uparrow}{\tfrac{3}{4}}}$$

We say 'the integers are <u>not</u> closed under scalar multiplication by $\mathbb{R}$'.

Sometimes we have vector spaces with 'strange' operations.

Non-example $V = \mathbb{R}^2$, with vector addition $\oplus$ and scalar multiplication $\odot$ defined by

$$\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ bd \end{bmatrix},$$

and

$$c \odot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix}.$$

Which axioms fail? (It is commutative, associative, has a zero $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, ...
but it doesn't have negatives, and
$c \odot (u \oplus v) \neq (c \odot u) \oplus (c \oplus v)$ in general.)

Example $V = \mathbb{R}$, with vector addition $\oplus$ and scalar multiplication $\odot$ defined by

$$a \oplus b = ab \quad \text{for } a, b \in \mathbb{R}$$
$$c \odot b = b^c \quad \text{for } b, c \in \mathbb{R}.$$

The zero vector is $1 \in \mathbb{R}$, and the negative of a is a^{-1}!

What on earth is going on here?!

Theorem

- There is a unique vector 0 such that $0+u=u$ for all u.
- There is a unique vector $-u$ such that $u+(-u)=0$.

  (That is, if $u+v=0$, and $u+w=0$, then $v=w$.)
- $\underbrace{0}_{\text{the scalar zero}} u = \underbrace{0}_{\text{the vector zero}}$ for all $u \in V$.
- $c0=0$
- $(-1)u=-u$.

(See Exercises 4.1.25–29 in Lay for the proofs.)

Subspaces

Sometimes one vector space sits inside another.

For example, $\mathbb{P}_2 \subset \mathbb{P}_4$: we can write a degree two polynomial

as $$p(t) = a_0 + a_1 t + a_2 t^2 + 0t^3 + 0t^4.$$

Definition

A subspace of a vector space V is a subset H of V such that

- the zero vector is in H
- whenever $u, v \in H$, $u+v$ is also in H
- if $u \in H$, then cu is in H for all $c \in \mathbb{R}$.

Theorem A subspace is a vector space in its own right.

That is, all the axioms automatically hold, as long as you've checked the subset contains zero and is closed under addition and scalar multiplication.

Examples

Let $W = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$ be the set of 2×2 symmetric matrices.

Then W is a subspace of $M_{2\times 2}$.

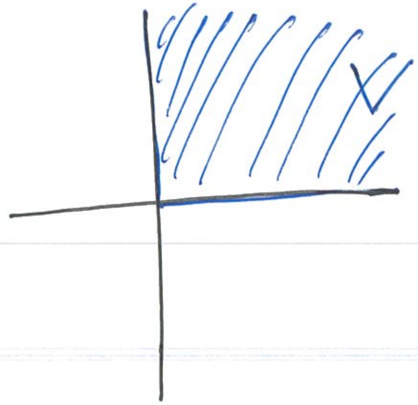
- The zero vector is symmetric: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
- If A and B are symmetric, then

$$(A+B)^T = A^T + B^T = A+B,$$

so $A+B \in W$, too.

- Similarly $(cA)^T = cA^T = cA$, so W is closed under scalar multiplication.

Non-example Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0,\ y \geq 0 \right\}$.



Is this a vector space? It is a subset of $\mathbb{R}^2$.

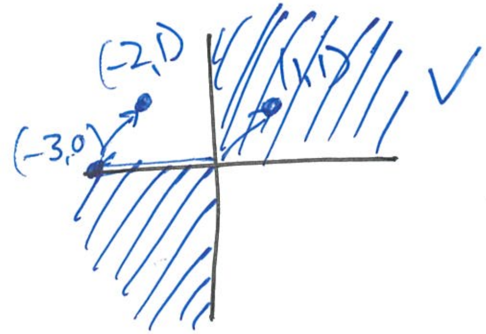
It contains the zero vector for $\mathbb{R}^2$.

It is closed under ~~scalar mu~~ vector addition.

But it is not closed under scalar multiplication

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in V, \quad \text{but} \quad (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin V.$$

Non-example Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$



It contains the zero vector.

It is closed under scalar multiplication:

if $xy \geq 0$ is $\begin{bmatrix} cx \\ cy \end{bmatrix} \in V$? Yes: $(cx)(cy) = c^2xy \geq 0$.

It is not closed under vector addition. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix} \in V$,

$$\text{but} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \notin V.$$

Non-example $U = \{ p \in \mathbb{P}_2 : p(1) = 2 \}$

Example $V = \{ p \in \mathbb{P}_2 : p(1) = 0 \}$.

Definition

Given a set of vectors $S = \{v_1, \ldots, v_p\}$ in a vector space V, the span of S is the set of all vectors which can be written as a linear combination of vectors in S.

$$\mathrm{Span}(S) = \{c_1 v_1 + c_2 v_2 + \cdots + c_p v_p : c_i \in \mathbb{R}\},$$

Theorem Span(S) is a subspace of V.

In fact, you can think of Span(S) as the smallest vector space containing all the v_i.

Example

Show that

$$W = \left\{ \begin{bmatrix} 4a-2b \\ a+b+c \\ 0 \\ -2c-6a \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^4$.

Observe that

$$W = \left\{ a\begin{bmatrix} 4 \\ 1 \\ 0 \\ -6 \end{bmatrix} + b\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} 4 \\ -1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

and spans are automatically subspaces.