Last time

We defined an abstract vector space as a set of objects satisfying a list of axioms.

Examples: $\mathbb{R}^n$, polynomials of degree at most d, the set of $n \times n$ matrices.

We also defined a subspace: a subset of a vector space which

- contains zero
- is closed under vector addition
- is closed under scalar multiplication.

Example: the 2×2 symmetric matrices form a vector subspace of the set of all 2×2 matrices.

Recall matrices and linear equations.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

The equation $Ax = b$ can be written as

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \end{aligned}$$

and we find solutions by row-reducing the ~~ag~~ augmented matrix

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \end{array}\right]$$

to reduced echelon form.

The null space of an $m \times n$ matrix A is the set of solutions to $Ax = 0$.

$$\mathrm{Nul}\, A = \{ x \in \mathbb{R}^n \mid Ax = 0 \}.$$

Example

Let $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}$.

What is $\mathrm{Nul}\, A$?

$Ax = 0 \Rightarrow x_2 = 3x_3,\ x_1 = -4x_3,$ so

$$\mathrm{Nul}\, A = \mathrm{span} \left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

<u>Theorem</u> the null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

Proof ~~the~~ $\operatorname{Nul} A$ is a subset of $\mathbb{R}^n$, so we need to check the three conditions.

- $0 \in \operatorname{Nul} A$, because $A0 = 0$.
- if $u, v \in \operatorname{Nul} A$, then $A(u+v) = Au + Av = 0 + 0 = 0$. So $u+v \in \operatorname{Nul} A$.
- if $u \in \operatorname{Nul} A$, $c \in \mathbb{R}$, then $A(cu) = c(Au) = c0 = 0$, so $cu \in \operatorname{Nul} A$.

Example

Let $W =$

$$\left\{ \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} \middle| \begin{array}{l} 3s - 4u = 5r + t \\ 3r + 2s - 5t = 4u \end{array} \right\}$$

Show that W is a subspace of $\mathbb{R}^4$.

These equations are equivalent to

$$-5r + 3s - t - 4u = 0$$
$$3r + 2s - 5t - 4u = 0,$$

which is same as saying

$$\begin{bmatrix} -5 & 3 & -1 & -4 \\ 3 & 2 & -5 & -4 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} = 0.$$

Thus $W = \text{Nul} \begin{bmatrix} -5 & 3 & -1 & -4 \\ 3 & 2 & -5 & -4 \end{bmatrix}$,

and so is a subspace of $\mathbb{R}^4$, by the theorem.

The column space of an $m \times n$ matrix A is the set of all linear combinations of columns of A.

If $A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$, then $\mathrm{Col}\, A = \mathrm{span}\{a_1, a_2, \ldots, a_n\}$.

Theorem The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

An equivalent description is

$$\mathrm{Col}\, A = \{ Ax \mid x \in \mathbb{R}^n \}.$$

Example

Let $u = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$ and

$$A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Does u lie in the column space of A?

We need to answer: does $Ax = u$ have a solution?

We do some row reduction:

$$\left[\begin{array}{cc|c} 3 & 1 & 5 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 4 & 8 \\ 0 & 1 & 2 \\ 1 & -1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

This says the system $Ax = u$ is consistent,
so u can be written as a linear combo of
columns of A,

$\Longrightarrow$ u is in $\operatorname{Col} A$.

For an $m \times n$ matrix A:

Nul A	Col A
Nul A is a subspace of $\mathbb{R}^n$	Col A is a subspace of $\mathbb{R}^m$
$v \in \text{Nul} A \iff Av = 0$	$v \in \text{Col} A \iff$ the equation $Ax = v$ is consistent
Nul $A = \{0\}$ if and only if the equation $Ax = 0$ has only the trivial solution.	Col $A = \mathbb{R}^m$ if and only if $Ax = b$ is consistent for every $b \in \mathbb{R}^m$.

A linear transformation from a vector space V to a vector space W is a function $T: V \to W$ such that

- $T(u+v) = T(u) + T(v)$ for all $u, v \in V$
- $T(cu) = cT(u)$ for all $c \in \mathbb{R}$, $u \in V$.

Example Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 4 \end{bmatrix}$. The function $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T_A(x) = Ax$ is a linear transformation.

Example The function $T: \mathbb{P}_2 \to \mathbb{R}$ defined by

$$T(p) = p(7)$$

is a linear transformation.

- The **kernel** of a linear transformation $T: V \to W$ is the set of all vectors $u \in V$ so $T(u) = 0$.

$$\ker T = \{u \in V \mid T(u) = 0\}.$$

It is a subspace of V.

If $\ker T = \{0\}$, we say **T is one-to-one.**

- The **range** of a linear transformation $T: V \to W$ is the set of all vectors of the form $T(u)$ where $u \in V$.

$$\operatorname{range} T = \{w \in W \mid w = T(u) \text{ for some } u \in V\}.$$

It is a subspace of W.

We say **T is onto** if $\operatorname{range} T = W$.

If A is an $m \times n$ matrix we can define

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

by $T_A(x) = Ax$. T_A is a linear transformation.

$$\ker T_A = \operatorname{Nul} A$$

$$\operatorname{range} T_A = \operatorname{Col} A.$$

Example

Let $D : \mathbb{P}_2 \longrightarrow \mathbb{P}_1$ be the differential operator

$$D(p)(x) = p'(x).$$

Find the kernel and range of D.

Exercise

Show that D is a linear transformation.

$$\ker D = \{ p \in \mathbb{P}_2 \mid p' = 0 \}$$

$$= \{ a_0 + a_1 t + a_2 t^2 \mid \underbrace{a_1 + 2a_2 t = 0} \}$$

This isn't an equation in t! don't solve and say $t = -\frac{a_1}{2a_2}$!!!

This is an equation in $\mathbb{P}_1$

$$= \{ a_0 + a_1 t + a_2 t^2 \mid a_1 = a_2 = 0 \}$$

$$= \{ a_0 \mid a_0 \in \mathbb{R} \}$$

$$\operatorname{range} D = \{ p'(x) \mid p(x) \in \mathbb{P}_2 \}$$

$$= \{ a_1 + 2a_2 t \mid a_0 + a_1 t + a_2 t^2 \in \mathbb{P}_2 \}$$

$$= \{ a_1 + 2a_2 t \mid a_1, a_2 \in \mathbb{R} \}.$$

Example

Define

$S : \mathbb{P}_2 \to \mathbb{R}^2$

by

$S(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.

What is $\ker S$?

What is range S?

$$\ker S = \{a_0 + a_1 t + a_2 t^2 \mid S(a_0 + a_1 t + a_2 t^2) = 0\}$$

$$= \left\{a_0 + a_1 t + a_2 t^2 \;\middle|\; \begin{bmatrix} a_0 \\ a_0 + a_1 + a_2 \end{bmatrix} = 0\right\}$$

$$= \left\{a_0 + a_1 t + a_2 t^2 \;\middle|\; \begin{matrix} a_0 = 0 \\ a_1 = -a_2 \end{matrix}\right\}$$

$$= \{a_1(t - t^2)\}$$

$$= \operatorname{span}\{t - t^2\}$$

$$= \{a + bt \mid a, b \in \mathbb{R}\}$$

$$= \mathbb{P}_1$$

Is D one-to-one?

No!

Is D onto?

Yes!

Example

Let $S: \mathbb{P}_1 \to \mathbb{R}$ be the linear transformation

$$S(p) = \int_0^1 p(x)\,dx$$

What are the kernel and range of S?