A set of vectors $\{v_1, \ldots, v_p\}$ in a vector space V is

linearly independent

if the only solution to the vector equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$$

is the trivial solution

$$c_1 = 0,\ c_2 = 0,\ \ldots,\ c_p = 0.$$

If there's a non-trivial solution i.e. not all the c_i zero, then $\{v_1, \ldots, v_p\}$ is linearly dependent.

Example

Show $\{2x+3, 4x^2, 1+x\}$ is linearly independent in $\mathbb{P}_2$

degree at most 2 polynomials.

We look at an arbitrary linear combo, and see what it means if it's zero:

$$a(2x+3)+b(4x^2)+c(1+x)=0$$

$$\Updownarrow$$

$$(4b)x^2+(2a+c)x+(3a+c)=0$$

$$\Updownarrow$$

$$4b=0,\quad 2a+c=0,\quad 3a+c=0$$

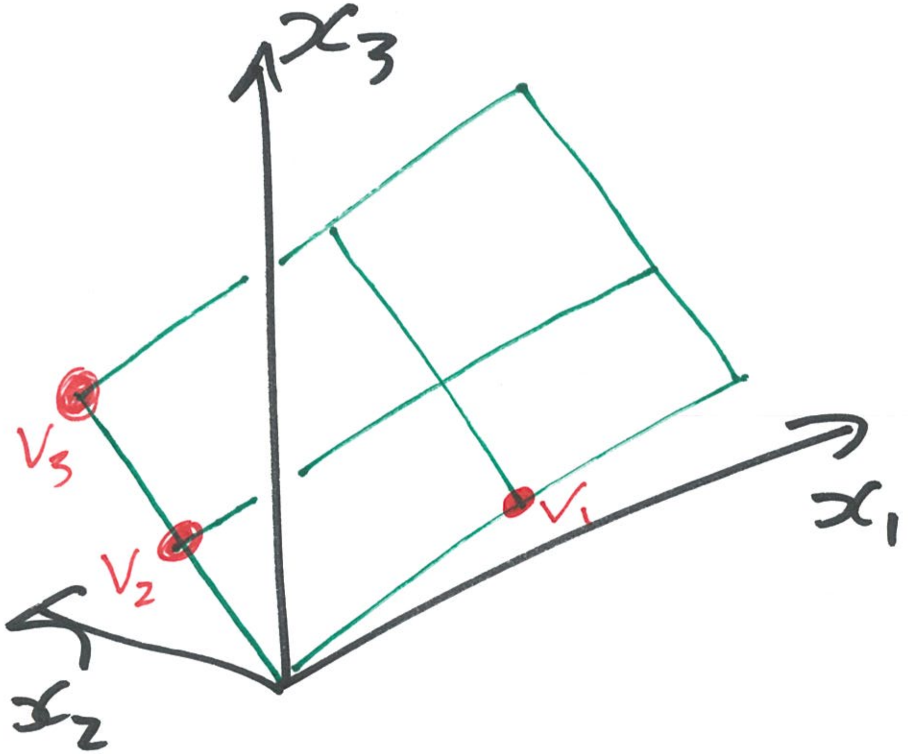
$$\Updownarrow$$

$$a=0 \quad b=0 \quad c=0$$

So the only solution is the trivial one, so the set is linearly independent!

Example

Consider the plane H



Which is true?

- $H = \text{span}\{v_1, v_2\}$
- $H = \text{span}\{v_1, v_3\}$
- $H = \text{span}\{v_2, v_3\}$
- $H = \text{span}\{v_1, v_2, v_3\}$.

Even this one!

All except this one!

A set of vectors

$$\mathcal{B} = \{v_1, \dots, v_p\}$$

in a vector space V is called

a basis

if

1) $\mathcal{B}$ is a linearly independent set

2) $V = \operatorname{span} \mathcal{B}$.

Say $v_2 = v_1 + 2v_3$

If $x = av_1 + bv_2 + cv_3$

$= av_1 + b(v_1 + 2v_3) + cv_3$

$= (a+b)v_1 + (2b+c)v_3.$

The spanning set theorem.

Suppose $S = \{v_1, \dots, v_p\}$ is a subset of V, and let $H = \operatorname{span} S$.

Then

1) If is a vector v_k in S is a linear combo of the other vectors in S, then the set

$$S \setminus \{v_k\} \longleftarrow \text{"S with } v_k \text{ removed"}$$

still spans H.

2) Some subset of S is a basis for H.

Example

Find a basis for $\mathbb{P}_2$ which is a subset of

$S = \{1, x, 1+x, x+3, x^2\}$.

First, does S even span? (Yes!)

By the spanning set theorem, we can throw any vector which is a linear combo of others, without changing the span.

$$\mathbb{P}_2 = \operatorname{span} S = \operatorname{span}\{1, x, 1+x, x+3, x^2\}$$
$$= \operatorname{span}\{1, x, x+3, x^2\}$$
$$= \operatorname{span}\{1, x, x^2\}$$

Now $\{1, x, x^2\}$ is linearly independent, so a basis.

(Other answers: $\{1, 1+x, x^2\}$

Example

Find a basis for the nullspace of

$$A = \begin{pmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution

Row reducing,

$$\begin{pmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_1 \to r_1 - 5r_2} \begin{pmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which corresponds to the equations

$$\begin{aligned} x_1 + \quad 6x_3 - 8x_4 + x_5 &= 0 \\ x_2 - 2x_3 + x_4 &= 0 \end{aligned}$$

with free variables x_3, x_4, x_5 and

$$x_1 = -6x_3 + 8x_4 - x_5$$

$$x_2 = 2x_3 - x_4$$

Writing that in vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -6x_3+8x_4-x_5 \\ -2x_3-x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= x_3\begin{pmatrix} -6 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

These vectors u, v, w form a spanning set for the nullspace of A!

They're independent, so a basis!

This is an algorithm for finding a basis for NulA.

Example

Find a basis for the column space of

$$A = \begin{pmatrix} 1 & 0 & 6 & -3 & 0 \\ 4 & 3 & 33 & -6 & 8 \\ 2 & -1 & 9 & -8 & 4 \\ -2 & 2 & -6 & 10 & -2 \end{pmatrix}$$

Theorem:
The pivot columns of a matrix form a basis of the column space.

Let's row reduce, obtaing

$$B = \begin{pmatrix} 1 & 0 & 6 & -3 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

When we row-reduce the same linear relations hold between columns before and after.

Write $A = \begin{pmatrix} | & | & | & | & | \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ | & | & | & | & | \end{pmatrix}$ $B = \begin{pmatrix} | & | & | & | & | \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ | & | & | & | & | \end{pmatrix}$.

Notice that $b_3 = 6b_1 + 3b_2$

$b_4 = -3b_1 + 2b_2$

we can verify $a_3 = 6a_1 + 3a_2$

$a_4 = -3a_1 + 2a_2$.

It's easy to see $\{b_1, b_2, b_5\}$ (the pivot columns)

form a basis for Col B.

$\Rightarrow \{a_1, a_2, a_5\}$ is a basis for Col A.

The <u>unique representation theorem</u>

Suppose $\mathcal{B} = \{v_1, \dots, v_p\}$ is a basis for V.
Then each $x \in V$ can be expressed
<u>uniquely</u> as

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

for some $c_i \in \mathbb{R}$.

We say these c_i are the coordinates of x relative to the basis $\mathcal{B}$ and write

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

Example

Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x+3, x^2\}$. Find the coordinates of

$$5 + 2x + x^2$$

in $\mathcal{B}$ and in $\mathcal{C}$.

$$5 + 2x + x^2 = 5(1) + 2(x) + 1(x^2)$$

so $[5 + 2x + x^2]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$

Similarly

$$5 + 2x + x^2 = -1(1) + 2(x+3) + 1(x^2)$$

so

$$[5 + 2x + x^2]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Proof

Suppose $\mathcal{B} = \{v_1, \dots, v_p\}$ is a basis for V.

Since a basis is a spanning set, it's possible to write

$$x = c_1 v_1 + \cdots + c_p v_p.$$

What if we have another way:

$$x = d_1 v_1 + \cdots + d_p v_p.$$

$$0 = x - x = (c_1 - d_1)v_1 + (c_2 - d_2)v_2 + \cdots + (c_p - d_p)v_p.$$

Since a basis is linearly independant, the only sol^n is the trivial one, so

$$c_1 = d_1, \; c_2 = d_2, \; \dots, \; c_p = d_p.$$

That, they were the same.