Overview

Last time we defined a *basis* of a vector space H:

Definition

The set $\{\mathbf{v_1}, \cdots, \mathbf{v_p}\}$ is a basis for H if

- $\{\mathbf{v_1}, \cdots, \mathbf{v_p}\}$ is linearly independent, and
- $\text{Span}\{\mathbf{v_1}, \cdots, \mathbf{v_p}\} = H$

We recalled algorithms (§2.8, §4.3) to find a basis for the null space and the column space of a matrix, and we stated the Unique Representation Theorem:

> *Given a basis for H, every vector in H can be a written as a linear combination of basis vectors in a unique way.*

The coefficients of this expression are the *coordinates* of the vector with respect to the basis.

Question

Given bases $\mathcal{B}$ and $\mathcal{C}$ for H, how are $[\mathbf{x}]_\mathcal{B}$ and $[\mathbf{x}]_\mathcal{C}$ related?

Coordinates

Theorem (The Unique Representation Theorem)

Suppose that $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for a vector space V. Then each $\mathbf{x} \in V$ has a unique expansion

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n \qquad (1)$$

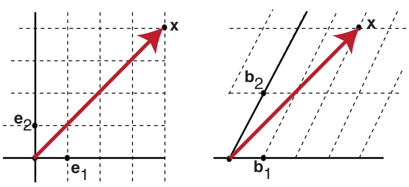
where $c_1, \dots, c_n$ are in $\mathbb{R}$.

We say that the c_i are the *coordinates* of $\mathbf{x}$ relative to the basis $\mathcal{B}$, and we write $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Coordinates give instructions for writing a given vector as a linear combination of basis vectors.

Different bases determine different coordinates...

Suppose $\mathcal{B} = \{\begin{bmatrix}1\\0\end{bmatrix}_\mathcal{E}, \begin{bmatrix}1\\2\end{bmatrix}_\mathcal{E}\}$, and as always, $\mathcal{E} = \{\begin{bmatrix}1\\0\end{bmatrix}_\mathcal{E}, \begin{bmatrix}0\\1\end{bmatrix}_\mathcal{E}\}$.



Standard graph   *B*-graph paper

$$\text{If } [\mathbf{x}]_\mathcal{B} = \begin{bmatrix}2\\2\end{bmatrix}, \quad \text{then} \quad \mathbf{x} = 2\mathbf{b}_1 + 2\mathbf{b}_2 = 2\begin{bmatrix}1\\0\end{bmatrix}_\mathcal{E} + 2\begin{bmatrix}1\\2\end{bmatrix}_\mathcal{E} = \begin{bmatrix}4\\4\end{bmatrix}_\mathcal{E}$$

$$\text{Similarly, } [\mathbf{x}]_\mathcal{E} = \begin{bmatrix}4\\4\end{bmatrix}, \quad \text{so} \quad \mathbf{x} = 4\mathbf{e}_1 + 4\mathbf{e}_2 = 4\begin{bmatrix}1\\0\end{bmatrix}_\mathcal{E} + 4\begin{bmatrix}0\\1\end{bmatrix}_\mathcal{E} = \begin{bmatrix}4\\4\end{bmatrix}_\mathcal{E}$$

...but some things stay the same

Even though we use different coordinates to describe the same point with respect to different bases, the structures we see in the vector space are independent of the chosen coordinates.

Definition

A one-to-one and onto linear transformation between vector spaces is an *isomorphism*. If there is an isomorphism $T : V_1 \to V_2$, we say that V_1 and V_2 are *isomorphic*.

Informally, we say that the vector space V is isomorphic to W if *every vector space calculation in V is accurately reproduced in W*, and vice versa.

For example, the property of a set of vectors being linearly independent doesn't depend on what coordinates they're written in.

Isomorphism

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $P : V \to \mathbb{R}^n$ defined by $P(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ is an isomorphism.

What does this theorem mean?
V and $\mathbb{R}^n$ are both vector spaces, and we're defining a specific map that takes vectors in V to vectors in $\mathbb{R}^n$. This map

- ...is a linear transformation
- ...is one-to-one (i.e., if $P(\mathbf{u}) = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$)
- ...is onto (for every $\mathbf{v} \in \mathbb{R}^n$, there's some $\mathbf{u} \in V$ with $P(\mathbf{u}) = \mathbf{v}$)

Every vector space with an n-element basis is isomorphic to $\mathbb{R}^n$.

Very Important Consequences

If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V then

- A set of vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in V spans V if and only if the set of the coordinate vectors $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}\}$ spans $\mathbb{R}^n$;
- A set of vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in V is linearly independent in V if and only if the set of the coordinate vectors $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$.
- An indexed set of vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in V is a basis for V if and only if the set of the coordinate vectors $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}\}$ is a basis for $\mathbb{R}^n$.

Theorem

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors is linearly dependent.

Theorem

If a vector space V has a basis consisting of n vectors, then every basis of V must consist of exactly n vectors.

That is, every basis for V has the same number of elements. This number is called the *dimension* of V and we'll study it more tomorrow.

Changing Coordinates (Lay §4.7)

When a basis $\mathcal{B}$ is chosen for V, the associated coordinate mapping onto $\mathbb{R}^n$ defines a coordinate system for V. Each $\mathbf{x} \in V$ is identified uniquely by its coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

In some applications, a problem is initially described by using a basis $\mathcal{B}$, but by choosing a different basis $\mathcal{C}$, the problem can be greatly simplified and easily solved.

We want to study the relationship between $[\mathbf{x}]_{\mathcal{B}}, [\mathbf{x}]_{\mathcal{C}}$ in $\mathbb{R}^n$ and the vector $\mathbf{x}$ in V. We'll try to solve this problem in 2 different ways.

Changing from $\mathcal{B}$ to $\mathcal{C}$ coordinates: Approach #1

Example 1

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose that

$$\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2. \tag{2}$$

Further, suppose that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for some vector $\mathbf{x}$ in V. What is $[\mathbf{x}]_{\mathcal{C}}$?

Let's try to solve this from the definitions of the objects:

Since $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ we have

$$\mathbf{x} = 2\mathbf{b}_1 + 3\mathbf{b}_2. \tag{3}$$

The coordinate mapping determined by $\mathcal{C}$ is a linear transformation, so we can apply it to equation (3):

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [2\mathbf{b}_1 + 3\mathbf{b}_2]_{\mathcal{C}} \\ &= 2[\mathbf{b}_1]_{\mathcal{C}} + 3[\mathbf{b}_2]_{\mathcal{C}} \end{aligned}$$

We can write this vector equation as a matrix equation:

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \tag{4}$$

Here the vector $[\mathbf{b}_i]_{\mathcal{C}}$ becomes the i^{th} column of the matrix.
This formula gives us $[\mathbf{x}]_{\mathcal{C}}$ once we know the columns of the matrix. But from equation (2) we get

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad \text{and} \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

So the solution is

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ -1 \end{bmatrix} \qquad \text{or} \\ [\mathbf{x}]_{\mathcal{C}} &= \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}} \end{aligned}$$

where $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$ is called the *change of coordinate matrix from basis $\mathcal{B}$ to $\mathcal{C}$.*

Note that from equation (4), we have

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix}$$

The argument used to derive the formula (4) can be generalised to give the following result.

Theorem (2)

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases for a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}}. \tag{5}$$

The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}. \tag{6}$$

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in Theorem 12 is called the **change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.**
Multiplication by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.

Of course,

$$[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} [\mathbf{x}]_{\mathcal{C}} ,$$

so that

$$[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}},$$

whence $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are inverses of each other.

Summary of Approach #1

> *The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$.*

Why is this true, and what's a good way to remember this?

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ are bases for a vector space V. What is $[\mathbf{b}_1]_{\mathcal{B}}$?

$$[\mathbf{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We have

$$[\mathbf{b}_1]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{b}_1]_{\mathcal{B}},$$

so the first column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ needs to be the vector for $\mathbf{b}_1$ in $\mathcal{C}$ coordinates.

Example

Example 2

Find the change of coordinates matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ for the bases

$$\mathcal{B} = \{1, x, x^2\} \quad \text{and} \quad \mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$$

of $\mathbb{P}_2$.

Notice that it's "easy" to write a vector in $\mathcal{C}$ in $\mathcal{B}$ coordinates.

$$[1+x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [x+x^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad [1+x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Example 3 (continued)

Find the change of choordinates matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ for the bases

$$\mathcal{B} = \{1, x, x^2\} \quad \text{and} \quad \mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$$

of $\mathbb{P}_2$.

Since we just showed

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

we have

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}.$$

Suppose now that we have a polynomial $p(x) = 1 + 2x - 3x^2$ and we want to find its coordinates relative to the $\mathcal{C}$ basis.
We have

$$[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

and so

$$\begin{aligned} [p]_{\mathcal{C}} &= \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [p]_{\mathcal{B}} \\ &= \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}. \end{aligned}$$

Changing from $\mathcal{B}$ to $\mathcal{C}$ coordinates: Approach #2

As we just saw, it's relatively easy to find a change of basis matrix from a standard basis (e.g., $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{1, x, x^2, x^3\}$) to a non-standard basis.

We can use this fact to find a change of basis matrix between two non-standard bases, too. Suppose that $\mathcal{E}$ is a standard basis and $\mathcal{B}$ and $\mathcal{C}$ are non-standard bases for some vector space.
To change from $\mathcal{B}$ to $\mathcal{C}$ coordinates, first change from $\mathcal{B}$ to $\mathcal{E}$ coordinates and then change from $\mathcal{E}$ to $\mathcal{C}$ coordinates:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\mathbf{x} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P} \left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\mathbf{x} \right).$$

Since this is true for all $\mathbf{x}$, we can write the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ as a product of two matrices which are easy to find:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}.$$

Example 4

Consider the bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

We want to find the change of coordinate matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ using the method described above.

We have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 7 & 2 \\ -2 & -1 \end{bmatrix}, \quad \underset{\mathcal{E}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -5 \\ -1 & 4 \end{bmatrix}$$

Hence

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}^{-1} \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \frac{1}{3}\begin{bmatrix} 2 & -5 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$$

Examples: Approach #1

Example 5

Consider the bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We want to find the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$, and from $\mathcal{C}$ to $\mathcal{B}$.

Solution The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ involves the $\mathcal{C}$-coordinate vectors of $\mathbf{b}_1$ and $\mathbf{b}_2$. Suppose that

$$[\mathbf{b}_1]_\mathcal{C} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad [\mathbf{b}_2]_\mathcal{C} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

From the definition

$$\mathbf{b}_1 = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$\mathbf{b}_2 = y_1\mathbf{c}_1 + y_2\mathbf{c}_2 = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

To solve these systems simultaneously we augment the coefficient matrix with $\mathbf{b}_1$ and $\mathbf{b}_2$ and row reduce:

$$\begin{aligned} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} &= \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 4 & 1 & 8 & -5 \end{array}\right] \\ &\xrightarrow{rref} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{array}\right]. \end{aligned} \tag{7}$$

This gives

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad \text{and} \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix},$$

and

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

You may notice that the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ already appeared in (7). This is because the first column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ results from row reducing $\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{b}_1 \end{bmatrix}$ to $\begin{bmatrix} I & \vdots & [\mathbf{b}_1]_{\mathcal{C}} \end{bmatrix}$, and similarly for the second column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. Thus

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} I & \vdots & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{bmatrix}.$$

Example 6

Consider the bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

We want to find the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$, and from $\mathcal{C}$ to $\mathcal{B}$.

We use the following relationship:

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} I & \vdots & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{bmatrix}.$$

Here

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|cc} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{array}\right].$$

This gives

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}.$$

Further

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}.$$

Example 7

In $M_{2\times 2}$ let $\mathcal{B}$ be the basis

$$\left\{E_{11} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, E_{21} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, E_{12} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, E_{22} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

and let $\mathcal{C}$ be the basis

$$\left\{A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, B = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, C = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, D = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right\}$$

We find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and verify that $[X]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[X]_{\mathcal{B}}$ for $X = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$.

Solution To solve this problem directly we must find the coordinate vectors of $\mathcal{B}$ with respect to $\mathcal{C}$.

This would usually involve solving a system of 4 linear equations of the form $E_{11} = aA + bB + cC + dD$ where we need to find a, b, c and d.

We can avoid that in this case since we can find the required coefficients by inspection:
Clearly $E_{11} = A, E_{21} = -B + C, E_{12} = -A + B$ and $E_{22} = -C + D$.

Thus

$$[E_{11}]_{\mathcal{C}} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, [E_{21}]_{\mathcal{C}} = \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, [E_{12}]_{\mathcal{C}} = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, [E_{22}]_{\mathcal{C}} = \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}.$$

From this we have

$$\begin{aligned}\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} &= \begin{bmatrix}[E_{11}]_{\mathcal{C}} & [E_{21}]_{\mathcal{C}} & [E_{12}]_{\mathcal{C}} & [E_{22}]_{\mathcal{C}}\end{bmatrix}\\ &= \begin{bmatrix}1 & 0 & -1 & 0\\0 & -1 & 1 & 0\\0 & 1 & 0 & -1\\0 & 0 & 0 & 1\end{bmatrix}\end{aligned}$$

For $X = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$,

$$X = 1E_{11} + 3E_{21} + 2E_{12} + 4E_{22}$$

and $[X]_{\mathcal{B}} = \begin{bmatrix}1\\3\\2\\4\end{bmatrix}$.

We now want to verify that $[X]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[X]_{\mathcal{B}}$ for $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. From our calculations

$$\begin{aligned} [X]_{\mathcal{C}} &= \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[X]_{\mathcal{B}} \\ &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}. \end{aligned}$$

This is the coordinate vector of X with respect to the basis $\mathcal{C}$.

We check this as follows:

Since $[X]_{\mathcal{C}} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$ this means that X should be given by $-A - B - C + 4D$:

$$\begin{aligned} -A - B - C + 4D &= -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = X \end{aligned}$$

as it should be.