Recall last time the unique representation theorem:

If $\mathcal{B} = \{v_1, v_2, \ldots, v_p\}$ is a basis for V,

each $x \in V$ can be expressed uniquely as

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

for some $c_i \in \mathbb{R}$

We say these c_i are the coordinates of x relative to the basis $\mathcal{B}$ and write

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}.$$

Different bases give different coordinates: ①

Say $\mathbb{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ and $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

the standard basis of $\mathbb{R}^2$

If $[x]_{\mathbb{B}} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then $x = 2b_1 + 2b_2 = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

so $[x]_{\mathcal{E}} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

but nothing <u>essential</u> changes about our vector space when we choose a different basis.

<u>Def^n</u>: A one-to-one and onto linear transformation between vector spaces is called an

<u>isomorphism</u>.

If there is an isomorphism

$T : V_1 \to V_2$

we say that the vector spaces

V_1 and V_2 are <u>isomorphic</u>.

Isomorphisms preserve structure.

Say $T: V_1 \to V_2$ is an isomorphism.

Then:

1) $a+b=c$ in V_1 $\iff$ $T(a)+T(b)=T(c)$ in V_2

2) $c \cdot x = y$ in V_1 $\iff$ $c \cdot T(x) = T(y)$ in V_2

3) $\{u_1, \dots, u_k\}$ span H in V_1 $\iff$ $\{T(u_1), \dots, T(u_2)\}$ span $T(H)$ in V_2

4) $\{u_1, \dots, u_k\}$ are linearly independent in V_1 $\iff$ $\{T(u_1), \dots, T(u_k)\}$ are linearly independent in V_2.

5) $\{u_1, \dots, u_k\}$ is a basis for V_1 $\iff$ $\{T(u_1), \dots, T(u_k)\}$ is a basis for V_2.

Theorem

Suppose $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis for a vector space V.

Then the coordinate mapping

$$P : V \longrightarrow \mathbb{R}^n$$

these are both vector spaces

defined by

$$P(x) = [x]_{\mathcal{B}}$$

and we're defining a particular linear map between them by this formula.

is an isomorphism.

We're claiming:

- it's really a linear transformation
- it's one-to-one: i.e. $[x]_{\mathcal{B}} = 0 \in \mathbb{R}^n$ only if $x = 0 \in V$
- it's onto: for every $v \in \mathbb{R}^n$, there's some $x \in V$ so $[x]_{\mathcal{B}} = v$.

This has many important consequences: ⑤

- any vector space with an n-element basis is isomorphic to $\mathbb{R}^n$

Say $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis for V

- a set of vectors $\{u_1, \dots, u_p\} \subset V$ spans V if and only if $\{[u_1]_{\mathcal{B}}, [u_2]_{\mathcal{B}}, \dots, [u_p]_{\mathcal{B}}\}$ spans $\mathbb{R}^n$.

- a set of vectors $\{u_1, \dots, u_p\} \subset V$ is linearly independent in V iff the coordinate vectors $\{[u_1]_{\mathcal{B}}, \dots, [u_p]_{\mathcal{B}}\}$ are linearly independent in $\mathbb{R}^n$.

- $\{u_1, \dots, u_p\}$ is a basis for $V \Longleftrightarrow \{[u_1]_{\mathcal{B}}, \dots, [u_p]_{\mathcal{B}}\}$ is a basis for $\mathbb{R}^n$.

Theorem

If a vector space V has a basis $\{u_1, \dots, u_n\}$ then

1) any set in V with more than n elements is linearly dependent
2) any set in V with fewer than n elements does not span V
3) any other basis for V also has exactly n elements.

Every basis for a vector space has the same number of elements.

We call this number the dimension of the vector space.

Changing coordinates

Say $\mathcal{B}$ and $\mathcal{C}$ are two bases for V.

How are $x \in V$, $[x]_{\mathcal{B}} \in \mathbb{R}^n$ and $[x]_{\mathcal{C}} \in \mathbb{R}^n$ related?

Example

$\mathcal{B} = \{b_1, b_2\}$, $\mathcal{C} = \{c_1, c_2\}$

are bases for V.

Suppose

$$b_1 = -c_1 + 4c_2$$

$$b_2 = 5c_1 - 3c_2,$$

and $[x]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

What is $[x]_{\mathcal{C}}$?

$[x]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ means

$$\begin{aligned} x &= 2b_1 + 3b_2 \\ &= 2(-c_1 + 4c_2) + 3(5c_1 - 3c_2) \\ &= (-2+15)c_1 + (8-9)c_2 \\ &= 13c_1 - c_2 \end{aligned}$$

so $[x]_{\mathcal{C}} = \begin{pmatrix} 13 \\ -1 \end{pmatrix}$.

Let's look at that again.

Since $x \mapsto [x]_{\mathcal{E}}$ is a linear transformation,

$$[x]_{\mathcal{E}} = [2b_1 + 3b_2]_{\mathcal{E}}$$

$$= 2[b_1]_{\mathcal{E}} + 3[b_2]_{\mathcal{E}}$$

$$= \begin{pmatrix} | & | \\ [b_1]_{\mathcal{E}} & [b_2]_{\mathcal{E}} \\ | & | \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

rewriting as a matrix equation.

the matrix with columns $[b_i]_{\mathcal{E}}$.

$$= \begin{pmatrix} -1 & 5 \\ 4 & -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Notice this was $[x]_{\mathcal{B}}$.

$$= \begin{pmatrix} 13 \\ -1 \end{pmatrix}$$

We just proved:

Change of coordinates theorem

Let $\mathcal{B} = \{b_1, \dots, b_n\}$ and $\mathcal{C} = \{c_1, \dots, c_n\}$ be bases for V.

Define an $n \times n$ matrix

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{pmatrix} | & | & & | \\ [b_1]_{\mathcal{C}} & [b_2]_{\mathcal{C}} & \cdots & [b_n]_{\mathcal{C}} \\ | & | & & | \end{pmatrix}.$$

Then $$[x]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [x]_{\mathcal{B}}$$

for any vector $x \in V$.

This matrix is called the change of coordinates matrix (from $\mathcal{B}$ to $\mathcal{C}$).

Observe $[x]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow \mathcal{E}}{P} [x]_{\mathcal{E}}$

$$= \underset{\mathcal{B}\leftarrow \mathcal{E}}{P} \; \underset{\mathcal{E}\leftarrow \mathcal{B}}{P} [x]_{\mathcal{B}},$$

so $\underset{\mathcal{B}\leftarrow \mathcal{E}}{P}$ and $\underset{\mathcal{E}\leftarrow \mathcal{B}}{P}$ are <u>inverses</u> of each other.

$$\boxed{\text{The columns of } \underset{\mathcal{E}\leftarrow \mathcal{B}}{P} \text{ are the } \mathcal{E}\text{-coordinates of the vectors in the basis } \mathcal{B}.}$$

How to remember? $[b_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$[b_1]_{\mathcal{E}} = \underset{\mathcal{E}\leftarrow \mathcal{B}}{P} [b_1]_{\mathcal{B}} = \underset{\mathcal{E}\leftarrow \mathcal{B}}{P} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \text{"the first column of } \underset{\mathcal{E}\leftarrow \mathcal{B}}{P}\text{"}.$$

Example

Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$

when

$\mathcal{B} = \{1, x, x^2\}$

$\mathcal{C} = \{1+x, x+x^2, 1+x^2\}$

are bases for $\mathbb{P}_2$.

One direction is <u>easy</u>.

$$[1+x]_{\mathcal{B}} = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad [x+x^2]_{\mathcal{B}} = \begin{pmatrix}0\\1\\1\end{pmatrix} \quad [1+x^2]_{\mathcal{B}} = \begin{pmatrix}1\\0\\1\end{pmatrix}.$$

Thus $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}$.

We can compute

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}^{-1} = \begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}^{-1} = \cdots = \begin{pmatrix}\frac12&\frac12&-\frac12\\-\frac12&\frac12&\frac12\\\frac12&-\frac12&\frac12\end{pmatrix}.$$

Let's sanity check. What is $[p]_{\mathcal{C}}$ when $p(x) = 1+2x-3x^2$?

$[p]_{\mathcal{B}} = \begin{pmatrix}1\\2\\-3\end{pmatrix}$, so $[p]_{\mathcal{C}} = \begin{pmatrix}\frac12&\frac12&-\frac12\\-\frac12&\frac12&\frac12\\\frac12&-\frac12&\frac12\end{pmatrix}\begin{pmatrix}1\\2\\-3\end{pmatrix} = \begin{pmatrix}3\\-1\\-2\end{pmatrix}$

and $3(1+x) - (x+x^2) - 2(1+x^2) = (3-2) + (3-1)x + (-1-2)x^2$
$= p(x)$.

Sometimes the best way to change bases is to go via a third basis (usually a 'standard' basis).

If $\mathcal{B}, \mathcal{C}$ and $\mathcal{E}$ are bases, then

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\ \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}^{-1}\ \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}.$$

__Example__

Consider

$\mathcal{B} = \left\{ \begin{pmatrix} 7 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$

$\mathcal{C} = \left\{ \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right\}$

What is $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$?

Well, $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 7 & 2 \\ -2 & -1 \end{pmatrix}$, $\underset{\mathcal{E}\leftarrow\mathcal{C}}{P} = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$, so

$$\underset{\mathcal{C}\leftarrow\mathcal{E}}{P} = \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix}$$

and

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\ \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ -2 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 24 & 12 \\ -15 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 4 \\ -5 & -2 \end{pmatrix}.$$