Last time:

- $[x]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{E}}{P}[x]_{\mathcal{E}}$, where $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P} = \begin{bmatrix} | & | & & | \\ [c_1]_{\mathcal{B}} & [c_2]_{\mathcal{B}} & \cdots & [c_n]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$

  (check: take $x = c_1$, $[x]_{\mathcal{E}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P}[x]_{\mathcal{E}} = [c_1]_{\mathcal{B}}$, as we want).

- Any two bases for a fixed vector space have the same number of elements.

  This number is called the <u>dimension</u> of the vector space.

Sometimes the best way to change bases is to go via a third basis (usually a 'standard' basis).

If $\mathcal{B}, \mathcal{C}$ and $\mathcal{E}$ are bases, then

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\ \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}^{-1}\ \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}.$$

Example

Consider

$\mathcal{B} = \left\{ \begin{pmatrix} 7 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$

$\mathcal{C} = \left\{ \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right\}$

What is $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$?

Well, $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 7 & 2 \\ -2 & -1 \end{pmatrix}$, $\underset{\mathcal{E}\leftarrow\mathcal{C}}{P} = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$, so

$$\underset{\mathcal{C}\leftarrow\mathcal{E}}{P} = \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix}$$

and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\ \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$

$$= \frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ -2 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 24 & 12 \\ -15 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 4 \\ -5 & -2 \end{pmatrix}.$$

Definition

If a vector space V is spanned by a finite set, we say V is finite dimensional.

Any two bases for V have the same number of elements, and this number is called the dimension of V.

The zero vector space $V = \{0\}$ has a basis $B = \{\}$, so its dimension is zero.

If V is not spanned by any finite set, we say V is infinite dimensional.

Examples

1. The standard basis for $\mathbb{R}^n$ has n vectors, so $\dim \mathbb{R}^n = n$.
2. The standard basis for $\mathbb{P}_3$ is $\{1, t, t^2, t^3\}$ so $\dim \mathbb{P}_3 = 4$.
3. The vector space of all continuous functions on the real line is infinite dimensional.

(Why? Can you show that the functions $\{1, x, x^2, \dots, x^n\}$ are linearly independent, whatever the value of n?
Why does this imply the claim?
Maybe it's easier to show $\{x, (x)(x-1), x(x-1)(x-2), \dots, x(x-1)(x-2)\cdots(x-n)\}$ is linearly independent...
Hint — look at a linear combination, and evaluate it at $x=1$, then at $x=2$, and so on.)

<u>Example</u> (Subspaces of $\mathbb{R}^3$)

0) The 0-dimensional subspace just contains $\begin{pmatrix}0\\0\\0\end{pmatrix}$.

1) If $u \neq 0$, then $\operatorname{span}\{u\}$ is a 1-dimensional subspace — a line through the origin.

2) If u and v are linearly independent in $\mathbb{R}^3$, then $\operatorname{Span}\{u,v\}$ is a 2-dimensional subspace — a plane through the origin.

3) The only 3-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

Theorem

Let H be a subspace of a finite dimensional space V.
Then H is finite dimensional and $\dim H \leq \dim V$.

- Any spanning set in H can be shrunk to a basis for H.
- Any linearly independent set in H can be expanded to a basis for H.

Example $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ is linearly independent in $\mathbb{R}^3$.
Can you expand it to a basis for $\mathbb{R}^3$?

Solution Adding an vector <u>not in</u> $\text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ will suffice.

E.g. $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

The Basis Theorem

Let V be a p-dimensional vector space.

- Any linearly independent set of exactly p elements is a basis for V.

- Any spanning set with exactly p elements is a basis for V.

Example

Schrödinger's equation in quantum mechanics for the harmonic oscillator is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0,$$

where $n = 0, 1, 2, \ldots$

The Hermite polynomials

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = -2 + 4x^2$

$H_3(x) = -12x + 8x^3$

$\vdots$

are solutions.

Show they form a basis for $\mathbb{P}_3$.

Writing their coordinates in the standard basis, we get

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -12 \\ 0 \\ 8 \end{pmatrix}.$$

Why are these linearly independent?

Since $\{H_0(x), H_1(x), H_2(x), H_3(x)\}$ is a 4-element linearly independent set, and $\dim \mathbb{P}_3 = 4$, this set must be a basis.

(And we <u>don't</u> need to check by hand that it spans.)

Recall from Monday:

- dim Nul A is the number of free variables in the equation $Ax = 0$.
- dim Col A is the number of pivot columns in the row-reduced form of A (and the corresponding columns of A itself form a basis of Col A).

<u>Example</u>

What are dim Nul A and dim Col A for

$$A = \begin{pmatrix} 1 & -6 & 9 & 10 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 3 pivots and 2 free variables, so $\dim(\text{Nul} A) = 2$ and $\dim(\text{Col} A) = 3$.

<u>Def^n</u>: the rank of a matrix is $\dim \operatorname{Col} A$.

~~Now dim Col A~~

Let A be a matrix and B be its reduced row echelon form.

Now $\dim \operatorname{Col} A = \#$ of pivot columns of B

and $\dim \operatorname{Nul} A = \#$ of free variables in $Ax = 0$

$= \#$ of non-pivot columns of B.

<u>Theorem</u> (Rank-nullity theorem)

If A is an $m \times n$ matrix (i.e it has n columns)

then $\operatorname{rank} A + \dim \operatorname{Nul} A = n$.

<u>Def^n</u> The row space Row$A$ of an $m\times n$ matrix is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

<u>Theorem</u>
- Suppose matrices A and B are related by row operations. Then Row A = Row B.
- If B is in echelon form, then the non-zero rows of B form a basis for Row B.

<u>Theorem</u> $\dim \text{Row} A = \dim \text{Col} A$

(Proof: both are the number of pivots in the echelon form of A).

Thus $\text{rank} A = \dim \text{Row} A = \dim \text{Col} A$,

and $\text{rank} A = \text{rank} A^T$ for any matrix A.

Example

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots

so Row A has basis $\{(1\,2\,0\,5), (0\,0\,1\,-3)\}$

and Col A has ~~basis~~ basis $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\}$.

Can you find a basis for Nul A?

(Before you start, how big should it be?)

Example

Suppose all the solutions of some homogeneous system of 5 linear equations in 6 unknowns are multiples of one non-zero solution.

Does the corresponding inhomogeneous system always admit a solution, whatever the constant?

$Ax=b$ always have a solution if $\dim \operatorname{Col} A$ = number of rows of A

For us, A is a 5×6 matrix,

so we want to know,

does $\dim \operatorname{Col} A = 5$?

(equivalently, $\operatorname{Col} A = \mathbb{R}^5$?)

Now $\dim \operatorname{Nul} A = 1$, so by rank-nullity

$$\dim \operatorname{Col} A = 6-1 = 5.$$

So yes, $Ax=b$ has a solution, whatever the value of b.

<u>Theorem</u> (Invertible matrix theorem, continuing from Lay §2.3)

Let A be an $n \times n$ matrix.

Then the following are equivalent:

a) A is an invertible matrix.

m) The columns of A form a basis of $\mathbb{R}^n$

n) $\text{Col}\, A = \mathbb{R}^n$

o) $\dim \text{Col}\, A = n$

p) $\text{rank}\, A = n$

q) $\text{Nul}\, A = \{0\}$

r) $\dim \text{Nul}\, A = 0$.