Overview

Given two bases $\mathcal{B}$ and $\mathcal{C}$ for the same vector space, we saw yesterday how to find the change of coordinates matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ nd $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$. Such a matrix is always square, since every basis for a vector space V has the same number of elements. Today we'll focus on this number —the *dimension* of V— and explore some of its properties.

From Lay, §4.5, 4.6

Dimension

Definition

If a vector space V is spanned by a finite set, then V is said to be **finite dimensional**.

The **dimension** of V, (written $\dim V$), is the number of vectors in a basis for V.

The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.

If V is not spanned by a finite set, then V is said to be **infinite dimensional**.

Example 1

1. The standard basis for $\mathbb{R}^n$ contains n vectors, so $\dim \mathbb{R}^n = n$.
2. The standard basis for $\mathbb{P}_3$, which is $\{1, t, t^2, t^3\}$, shows that $\dim \mathbb{P}_3 = 4$.
3. The vector space of continuous functions on the real line is infinite dimensional.

Dimension and the coordinate mapping

Recall the theorem we saw yesterday:

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $P : V \to \mathbb{R}^n$ defined by $P(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ is an isomorphism.

(Recall that an isomorphism is a linear transformation that's both one-to-one and onto.)

This means that every vector space with an n-element basis is isomorphic to $\mathbb{R}^n$.

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This means that every vector space with an n-element basis is isomorphic to $\mathbb{R}^n$. We can now rephrase this theorem in new language:

Theorem

Any n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Dimensions of subspaces of $\mathbb{R}^3$

Example 2

- The **0 - dimensional subspace** contains only the zero vector $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

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- If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors in $\mathbb{R}^3$, then Span $\{\mathbf{u}, \mathbf{v}\}$ is a **2 - dimensional subspace**. These subspaces are **planes** through the origin.

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- If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent vectors in $\mathbb{R}^3$, then Span $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a **3 - dimensional subspace**. This subspace is $\mathbb{R}^3$ itself.

Theorem

Let H be a subspace of a finite dimensional vector space V. Then any linearly independent set in H can be expanded (if necessary) to form a basis for H.

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Also, H is finite dimensional and

$$\dim H \leq \dim V.$$

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Let $H = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$. Then H is a subspace of $\mathbb{R}^3$ and $\dim H < \dim \mathbb{R}^3$. Furthermore, we can expand the given spanning set for

$H \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ to

$$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

to form a basis for $\mathbb{R}^3$.

Question

Can you find another vector that you could have added to the spanning set for H to form a basis for $\mathbb{R}^3$?

When the dimension of a vector space or subspace is known, the search for a basis is simplified.

Theorem (The Basis Theorem)

Let V be a p-dimensional space, $p \geq 1$.

1. *Any linearly independent set of exactly p elements in V is a basis for V.*
2. *Any set of exactly p elements that spans V is a basis for V.*

Example 4

Schrödinger's equation is of fundamental importance in quantum mechanics. One of the first problems to solve is the one-dimensional equation for a simple quadratic potential, the so-called linear harmonic oscillator.

Analysing this leads to the equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

where $n = 0, 1, 2, ...$

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There are polynomial solutions, the *Hermite polynomials*. The first few are

$$\begin{array}{ll} H_0(x) = 1 & H_3(x) = -12x + 8x^3 \\ H_1(x) = 2x & H_4(x) = 12 - 48x^3 + 16x^4 \\ H_2(x) = -2 + 4x^2 & H_5(x) = 120x - 160x^3 + 32x^5 \end{array}$$

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We want to show that these polynomials form a basis for $\mathbb{P}_5$.

Writing the coordinate vectors relative to the standard basis for $\mathbb{P}_5$ we get

$$\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\4\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-12\\0\\8\\0\\0\end{bmatrix}, \begin{bmatrix}12\\0\\0\\-48\\16\\0\end{bmatrix}, \begin{bmatrix}0\\120\\0\\-160\\0\\32\end{bmatrix}.$$

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Since $\dim \mathbb{P}_5 = 6$ and there are 6 polynomials that are linearly independent, the Basis Theorem shows that they form a basis for $\mathbb{P}_5$.

The dimensions of Nul A and Col A

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1. To find a basis for Nul A, use elementary row operations to transform $[A\ \mathbf{0}]$ to an equivalent reduced row echelon form $[B\ \mathbf{0}]$. Use the row reduced echelon form to find a parametric form of the general solution to $A\mathbf{x} = \mathbf{0}$. If Nul $A \neq \{\mathbf{0}\}$, the vectors found in this parametric form of the general solution are automatically linearly independent and form a basis for Nul A.

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2. A basis for Col A is is formed from the pivot columns of A. The matrix B determines the pivot columns, but it is important to return to the matrix A.

Dimension of Nul A and Col A

The dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.
The dimension of Col A is the number of pivot columns in A.

Example 5

Given the matrix

$$A = \begin{bmatrix} 1 & -6 & 9 & 10 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

what are the dimensions of the null space and column space?

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what are the dimensions of the null space and column space?

There are three pivots and two free variables, so $\dim(\text{Nul } A) = 2$ and $\dim(\text{Col } A) = 3$.

Example 6

Given the matrix

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there are three pivots and no free variables, $\dim(\text{Nul } A) = 0$ and $\dim(\text{Col } A) = 3$.

The rank theorem

As before, let A be a matrix and let B be its reduced row echelon form

$$\dim \operatorname{Col} A = \# \text{ of pivots of } A = \# \text{ of pivot columns of } B$$

Definition

The **rank** of a matrix A is the dimension of the column space of A.

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Definition

The **rank** of a matrix A is the dimension of the column space of A.

$$\dim \operatorname{Nul} A = \# \text{ of free variables of } B$$

$$= \# \text{ of non-pivot columns of } B.$$

Compare the two red boxes. What does this tell about the relationship between the dimensions of the null space and column space of matrix?

Theorem

If A is an $m \times n$ matrix, then

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Proof.

$$\left\{ \begin{array}{c} \text{number of} \\ \text{pivot columns} \end{array} \right\} + \left\{ \begin{array}{c} \text{number of} \\ \text{nonpivot columns} \end{array} \right\} = \left\{ \begin{array}{c} \text{number of} \\ \text{columns} \end{array} \right\}.$$

□

Examples

Example 7

If a 6×3 matrix A has rank 3, what can we say about $\dim \operatorname{Nul} A$, $\dim \operatorname{Col} A$ and $\operatorname{Rank} A$?

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- $\text{Rank } A + \dim \text{Nul } A = 3$.
- Since A only has three columns, and and all three are pivot columns, there are no free variables in the equation $A\mathbf{x} = \mathbf{0}$. Hence $\dim \text{Nul } A = 0$.
- $\dim \text{Col } A = \text{Rank } A = 3$.

The row space of a matrix

The null space and the column space are the fundamental subspaces associated to a matrix, but there's one other natural subspace to consider:

Definition

The *row space* $\mathrm{Row}\ A$ of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

Example 8

For the matrix A given by

$$A = \begin{bmatrix} 1 & -6 & 9 & 10 & -2 \\ 3 & 1 & 2 & -4 & 5 \\ -2 & 0 & -1 & 5 & 1 \\ 4 & -3 & 1 & 0 & 6 \end{bmatrix},$$

we can write

$$\begin{aligned} \mathbf{r}_1 &= [1, -6, 9, 10, -2] \\ \mathbf{r}_2 &= [3, 1, 2, -4, 5] \\ \mathbf{r}_3 &= [-2, 0, -1, 5, 1] \\ \mathbf{r}_4 &= [4, -3, 1, 0, 6 \end{aligned}$$

The row space of A is the subspace of $\mathbb{R}^5$ spanned by $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$.

(Note that we're writing the vectors $\mathbf{r}_i$ as rows, rather than columns, for convenience.)

A basis for $\mathrm{Row}\ B$

Theorem

Suppose a matrix B is obtained from a matrix A by row operations. Then $\mathrm{Row}\ A = \mathrm{Row}\ B$. If B is an echelon form of A, then the non-zero rows of B form a basis for $\mathrm{Row}\ B$.

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Compare this to our procedure for finding a basis for Col A. Notice that it's simpler: after row reducing, we don't need to return to the original matrix to find our basis!

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Proof.

If a matrix B is obtained from a matrix A by row operations, then the rows of B are linear combinations of those of A, so that $\mathrm{Row}\ B \subseteq \mathrm{Row}\ A$.
But row operations are reversible, which gives the reverse inclusion so that $\mathrm{Row}\ A = \mathrm{Row}\ B$.
In fact if B is an echelon form of A, then any non-zero row is linearly independent of the rows below it (because of the leading non-zero entry), and so the non-zero rows of B form a basis for $\mathrm{Row}\ B = \mathrm{Row}\ A$.

□

The Rank Theorem –Updated!

Theorem

For any $m \times n$ matrix A, Col A and Row A have the same dimension. This common dimension, the rank of A, is equal to the number of pivot positions in A and satisfies the equation

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This additional statement in this theorem follows from our process for finding bases for Row A and Col A:

Use row operations to replace A with its reduced row echelon form. Each pivot determines a vector (a column of A) in the basis for Col A and a vector (a row of B) in the basis for Row A.

Note also $$\text{Rank } A = \text{Rank } A^T.$$

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Example 10

If A is a 6×8 matrix, then the smallest possible dimension of $\text{Nul } A$ is 2.

Example 11

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Thus, $\{\mathbf{r}_1 = (1, 2, 0, 5),\ \mathbf{r}_2 = (0, 0, 1, -3)\}$ is a basis for $\mathrm{Row}\ A$.
(Note that these are rows of $rref(A)$, not rows of A.)

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Pivots are in columns 1 and 3 of $rref(A)$, so that $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}\ A$. (Note these are columns of A.)

Example 12

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \xrightarrow{ref} B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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The number of pivots in B is three, so $\dim \text{Col } A = 3$ and a basis for Col A is given by

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From B we can see that there are two free variables for the equation $A\mathbf{x} = \mathbf{0}$, so $\dim \operatorname{Nul} A = 2$. How would you find a basis for this subspace?

Applications to systems of equations

The rank theorem is a powerful tool for processing information about systems of linear equations.

Example 13

Suppose that the solutions of a homogeneous system of five linear equations in six unknowns are all multiples of one nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right hand side of the equations?

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Solution The hardest thing to figure out is

What is the question asking?

A non-homogeneous system of equations $A\mathbf{x} = \mathbf{b}$ always has a solution if and only if the dimension of the column space of the matrix A is the same as the length of the columns.

In this case if we think of the system as $A\mathbf{x} = \mathbf{b}$, then A is a 5×6 matrix, and the columns have length 5: each column is a vector in $\mathbb{R}^5$.
The question is asking

Do the columns span $\mathbb{R}^5$?

or equivalently,

Is the rank of the column space equal to 5?

First note that $\dim \mathrm{Nul}\ A = 1$. We use the equation:

$$\mathrm{Rank}\ A + \dim \mathrm{Nul}\ A = 6$$

to deduce that $\mathrm{Rank}\ A = 5$.
Hence the dimension of the column space of A is 5, $\mathrm{Col}\ A = \mathbb{R}^5$ and the system of non-homogeneous equations always has a solution.

Example 14

A homogeneous system of twelve linear equations in eight unknowns has two fixed solutions that are not multiples of each other, and all other solutions are linear combinations of these two solutions. Can the set of all solutions be described with fewer than twelve homogeneous linear equations? If so, how many?

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Considering the corresponding matrix system $A\mathbf{x} = \mathbf{0}$, the key points are

- A is a 12×8 matrix.
- $\dim \operatorname{Nul} A = 2$
- $\operatorname{Rank} A + \dim \operatorname{Nul} A = 8$
- What is the rank of A?
- How many equations are actually needed?

Example 15

Let $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$.

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Let $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$. The following are easily checked:

- Nul A is the z-axis.
- Row A is the xy-plane.
- Col A is the plane whose equation is $x + y = 0$.
- Nul A^T is the set of all multiples of $(1, 1, 0)$.
- Nul A and Row A are perpendicular to each other.
- Col A and Nul A^T are also perpendicular.

Theorem (Invertible Matrix Theorem ctd)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. *The columns of A form a basis of $\mathbb{R}^n$.*

n. *$Col\ A = \mathbb{R}^n$.*

o. $\dim Col\ A = n$.

p. *$Rank\ A = n$.*

q. *$Nul\ A = \{\mathbf{0}\}$.*

r. $\dim Nul\ A = 0$.

(The numbering continues the statement of the Invertible Matrix Theorem from Lay §2.3.)

Summary

1. Every basis for V has the same number of elements. This number is called the *dimension* of V.
2. If V is n-dimensional, V is isomorphic to $\mathbb{R}^n$.
3. A linearly independent list of vectors in V can be extended to a basis for V.
4. If the dimension of V is n, any linearly independent list of n vectors is a basis for V.
5. If the dimension of V is n, any spanning set of n vectors is a basis for V.