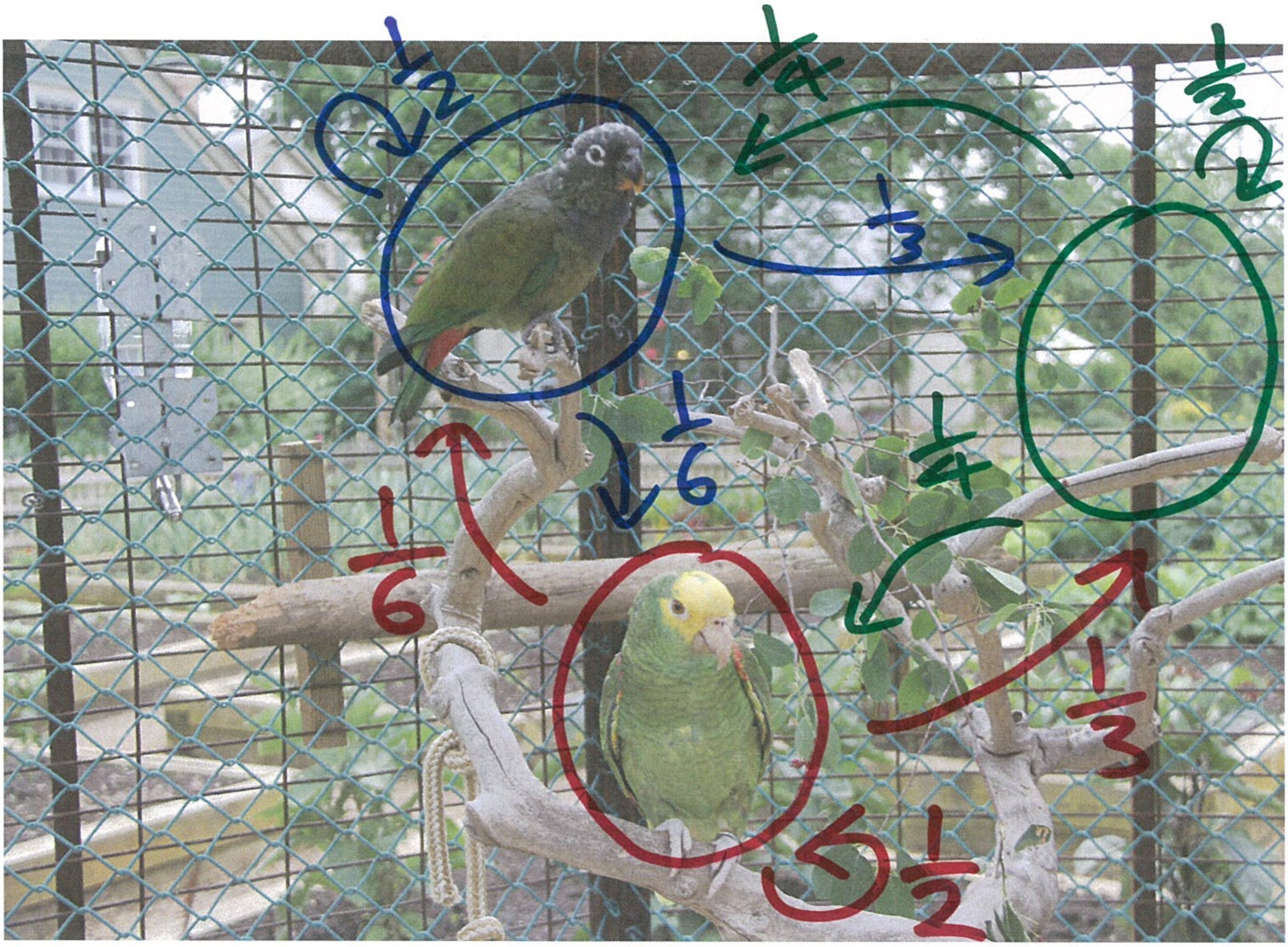


Let $p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ denote the probabilities that a bird is sitting at each site at some moment.

We should have $p_1 + p_2 + p_3 = 1$.

What are the probabilities at the next time step?

We can encode these probabilities in a matrix:

$$T = \begin{array}{cccc} \text{site 1} & \text{site 2} & \text{site 3} & \\ \left(\frac{1}{2} \right. & \frac{1}{4} & \left. \frac{1}{6} \right) & \text{site 1} \\ \left(\frac{1}{3} \right. & \frac{1}{2} & \left. \frac{1}{3} \right) & \text{site 2} \\ \left(\frac{1}{6} \right. & \frac{1}{4} & \left. \frac{1}{3} \right) & \text{site 3} \end{array}$$

The probability distribution at the next time step is just Tp.

Example $p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (i.e. we've just seen the bird sitting at site 1.)

$$Tp = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}$$ What about the next step?

$$TTp = T\begin{pmatrix}\frac{1}{2}\\ \frac{1}{3}\\ \frac{1}{6}\end{pmatrix} = \begin{pmatrix}13/36\\ 7/18\\ 1/4\end{pmatrix}$$

A vector $p=\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ with non-negative entries that add up to <u>one</u> is called a **probability vector.**

A square matrix, where each column is a probability vector is called a **stochastic matrix.**

<u>Exercise</u>: if A and B are stochastic matrices, show that AB is as well.

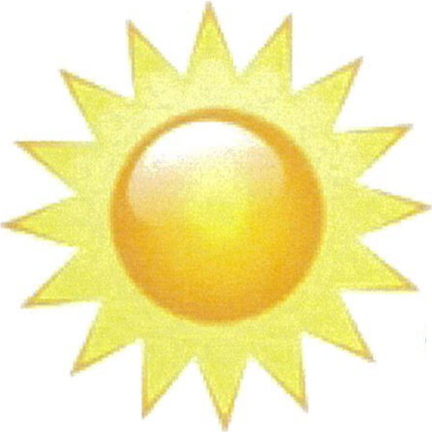
A **finite Markov chain** is a sequence of probability vectors $x_0, x_1, x_2, \ldots$ and a stochastic matrix T, such that

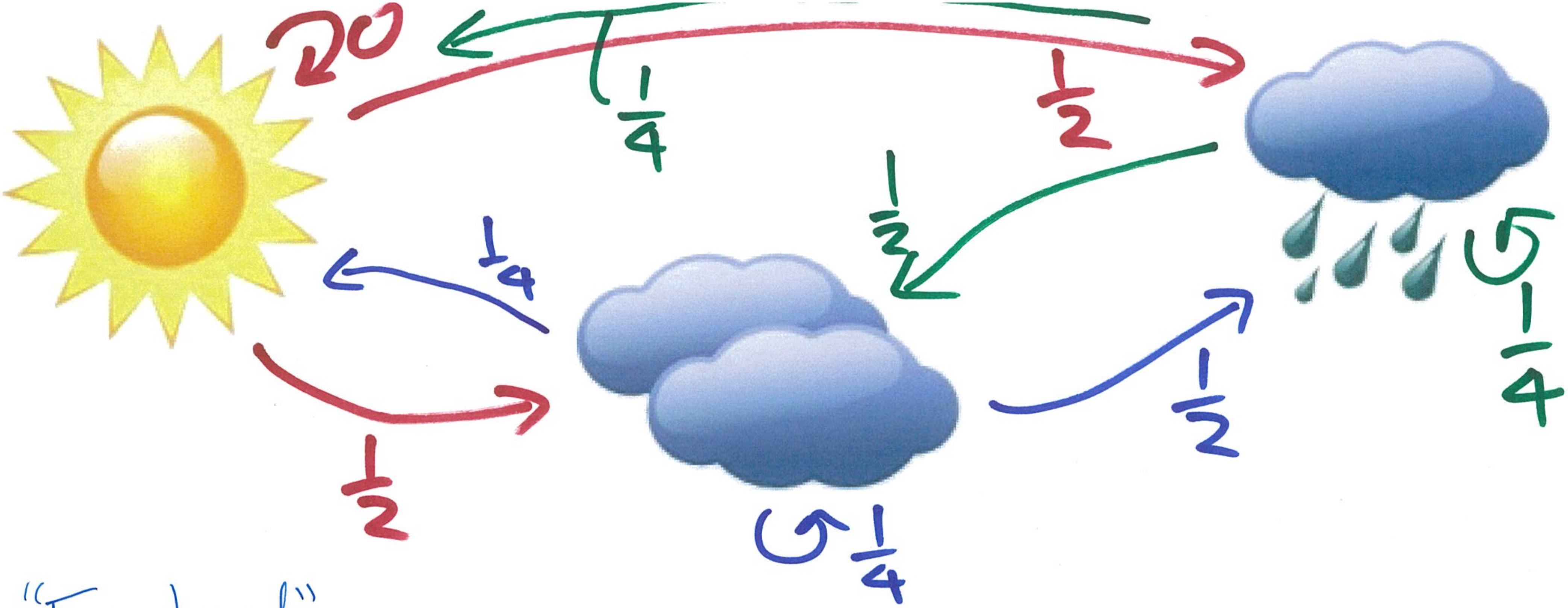
$$x_1 = Tx_0, \quad x_2 = Tx_1, \quad x_3 = Tx_2, \quad \ldots$$

The vector x_k is called a <u>state vector</u>.

When can we model a system using a finite Markov chain?

- at each time step, there are only finitely many possible states of the system.
- the state of the system <u>only</u> depends (perhaps probabilistically) on the state of the system at the immediately preceding time step.





"England"

$$T = \begin{array}{cc} \begin{array}{ccc} \text{fine} & \text{cloudy} & \text{rain} \end{array} & \\ \left(\begin{array}{ccc} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{array}\right) & \begin{array}{c} \text{fine} \\ \text{cloudy} \\ \text{rain} \end{array} \end{array}$$

This is our stochastic matrix modelling the weather.

Suppose it's raining on day 0. $x_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$x_1 = Tx_0 = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, \quad x_2 = Tx_1 = \begin{pmatrix} 3/16 \\ 3/8 \\ 7/16 \end{pmatrix}$$

What can we say about the distant future?

<u>Example</u>

$$x_7 = TTTTTTTx_0$$

$$= \begin{pmatrix} 0.2000122\ldots \\ 0.40002\ldots \\ 0.39996\ldots \end{pmatrix}$$

$$x_{15} = \begin{pmatrix} 0.2000000002 \\ 0.4000000003 \\ 0.3999999994 \end{pmatrix}$$

If T is a stochastic matrix, then a

steady state vector for T

is a probability vector q such that $Tq = q$.

<u>Theorem</u> Every stochastic matrix has a steady state vector.

To find one, we want to solve $Tx = x$,

or equivalently calculate $\mathrm{Nul}(T - I)$.

Example

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{2}{20} + \frac{2}{20} = \frac{1}{5} \\ \frac{1}{10} + \frac{2}{20} + \frac{2}{10} = \frac{2}{5} \\ \dots = \frac{2}{5} \end{pmatrix}$$

So $\begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{2}{5} \end{pmatrix}$ is a steady state vector.

Example For any 2×2 stochastic matrix with all nonzero entries, there is a unique steady state vector.

$$T = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix}$$

so $T - I = \begin{pmatrix} -p & q \\ p & -q \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} -p & q \\ 0 & 0 \end{pmatrix}$

so $\operatorname{Nul}(T-I) = \operatorname{span}\{q, p\}$,

and the only probability vector in there is

$$x = \frac{1}{p+q}\begin{pmatrix} q \\ p \end{pmatrix}$$

If $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, there is not a unique steady state vector, as $Tq = q$ for any probability vector.

When is the steady state vector unique?

Defⁿ A stochastic matrix is **regular** if some matrix power T^k has all entries strictly positive.

Example In "England", $T = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

but $T^2 = \begin{pmatrix} \frac{1}{4} & \frac{3}{16} & \frac{3}{16} \\ \frac{3}{8} & \frac{7}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{7}{16} \end{pmatrix}$, so T is regular.

Non-example $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and so on.

Theorem

If T is a regular stochastic matrix, T has a unique steady state vector q, and that vector has all entries strictly positive.

Moreover, for any initial probability vector x_0 the Markov chain $x_{k+1} = Tx_k$ converges: $\lim_{k \to \infty} x_k = q$.