Today: eigenvalues and eigenvectors

Recall time we considered a Markov chain modelling the weather in "England". $x_{k+1} = T x_k$ where

today is:

$$
T = \begin{array}{c c}
\begin{array}{ccc} \text{fine} & \text{cloudy} & \text{rainy} \end{array} & \\
\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} & \begin{array}{l} \text{fine} \\ \text{cloudy} \\ \text{rainy} \end{array}
\end{array}
$$

tomorrow

And I claimed that $T^7 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} = \begin{pmatrix} 0.20001\ldots \\ 0.40002\ldots \\ 0.39996\ldots \end{pmatrix}$

"It rained today, what's the weather next week?"

How can we do that calculation, without lots of matrix multiplication? What about computing T^{1000}?

Analyzing the eigenvalues and eigenvectors of a matrix will let us do very efficient matrix computations.

- and completely describe the behaviour of Markov chains, as well as more general linear dynamic systems.

- and show us when two linear transformations

$$T_1 : V \to V \text{ and } T_2 : V \to V$$

are "essentially the same".

<u>Def^n</u> an eigenvector for an $n \times n$ matrix A
is a <u>nonzero</u> vector x such that
$Ax = \lambda x$ for some scalar λ.

(That is, Ax points in the same direction as x.)

<u>Def^n</u> an eigenvalue for an $n \times n$ matrix A
is a scalar λ such that
$$Ax = \lambda x$$
has a non-zero solution.

When $x \neq 0$ satisfies $Ax = \lambda x$, we say:
"x is an eigenvector for A with eigenvalue λ"

<u>Example</u> Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

Any $\begin{pmatrix} x \\ 0 \end{pmatrix}$, for $x \neq 0$, is an eigenvector with eigenvalue 3:

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 3x \\ 0 \end{pmatrix} = 3\begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Any $\begin{pmatrix} 0 \\ y \end{pmatrix}$, for $y \neq 0$, is an eigenvector with eigenvalue 2.

Example Usually it's less obvious!

Let $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Then any $\begin{pmatrix} x \\ x \end{pmatrix}$, for $x \neq 0$, is an eigenvector with eigenvalue 2:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 2x \\ 2x \end{pmatrix}$$

Any $\begin{pmatrix} x \\ -x \end{pmatrix}$, for $x \neq 0$, is a eigenvector with eigenvalue 0:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} x \\ -x \end{pmatrix}.$$

An <u>eigenvalue</u> may be zero, but an <u>eigenvector</u> must be nonzero.

If we know (or suspect) λ is an eigenvalue for A, we find the corresponding eigenvectors by solving $(A - \lambda I)x = 0$.

That is, we find $\mathrm{Nul}(A - \lambda I)$.

<u>Def</u>

Def^{n} The λ-eigenspace of a square matrix A is $E_\lambda = \mathrm{Nul}(A - \lambda I)$.

<u>Example</u> Consider $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ again.

$$E_2 = \mathrm{Nul}(B - 2I) = \mathrm{Nul}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

$$E_0 = \mathrm{Nul}(B - 0I) = \mathrm{Nul}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}.$$

You can always double check an eigenvector — just multiply by the original matrix, and verify you get a scalar multiple.

Theorem The eigenvalues of a triangular matrix are the entries on the diagonal.

Proof (just for the 3×3 upper triangular case)

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{pmatrix}$

λ is an eigenvalue of A iff $(A-\lambda I)x = 0$ has a free variable, (i.e there's not a pivot in ~~any~~ every column of the rref).

Thus λ is an eigenvalue for A iff

$\lambda = a_{11}$, $\lambda = a_{22}$, or $\lambda = a_{33}$.

An $n \times n$ matrix A has λ as an eigenvalue

$$\Updownarrow$$

$Ax = \lambda x$ has a non-zero solution

$$\Updownarrow$$

$$\mathrm{Nul}(A - \lambda I) \neq \{0\}$$

$$\Updownarrow$$

$A - \lambda I$ is <u>not invertible</u>

<u>Example</u> 0 is an eigenvalue of $A \iff A$ is not invertible.

Next week we'll use this to ~~?~~ systematically find all eigenvalues.

Example Consider $A = \begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{pmatrix}$.

Find a basis for the 3-eigenspace.

$$A - 3I = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector with eigenvalue $3 \iff x + 2y + 3z = 0$.

$$E_3 = \left\{ \begin{pmatrix} -2y - 3z \\ y \\ z \end{pmatrix} : y, z \in \mathbb{R} \right\} = \left\{ y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

So $\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for E_3.

Theorem Fix A, a square matrix.
Suppose $\{v_1, \dots, v_k\}$ are eigenvectors with *distinct* eigenvalues $\{\lambda_1, \dots, \lambda_k\}$.

Then $\{v_1, \dots, v_k\}$ is a linearly independent set.

Proof Suppose $c_1 v_1 + \dots + c_k v_k = 0$, with $\{c_i\}$ not all zero.
Apply $(A - \lambda_1 I)$ to obtain $\underbrace{c_1(\lambda_1 - \lambda_1)v_1}_{=\text{zero}} + c_2(\lambda_2 - \lambda_1)v_2 + \dots + c_k(\lambda_k - \lambda_1)v_k$.

Since $v_1 \neq 0$, at least one of $\{c_2, \dots, c_k\}$ is nonzero, so

$$c_2\underbrace{(\lambda_2 - \lambda_1)}_{\neq 0}v_2 + \dots + c_k\underbrace{(\lambda_k - \lambda_1)}_{\neq 0}v_k = 0,$$

and we've found a smaller linearly dependent set. Continue, until you reach the contradiction $c_k' v_k = 0$, for some $c_k' \neq 0$.