Eigenvectors and eigenvalues

Recall: An eigenvector for a square matrix A is a non-zero vector x so that $Ax = \lambda x$ for some scalar λ.

The scalar λ is an eigenvalue for A.

If we know an eigenvalue λ, we can find the corresponding eigenvectors: they are all the non-zero vectors in the λ-eigenspace:

$$E_\lambda = \{x \mid Ax = \lambda x\} = \mathrm{Nul}(A - \lambda I).$$

But how do we determine the eigenvalues?

Example Let $A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$.

Then $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector for the eigenvalue 2:

$$Av = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2v$$

Also, $u = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector for the eigenvalue -1:

$$Au = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} = -1 \cdot u.$$

Now if x is nonzero and $Ax = \lambda x$, then

$$(A - \lambda I)x = 0 \text{ has non-trivial solutions}$$

$$\Updownarrow$$

$$A - \lambda I \text{ is not invertible}$$

$$\Updownarrow$$

$$\det(A - \lambda I) = 0$$

So the eigenvalues of A are exactly the solutions for λ in $\det(A - \lambda I) = 0$.

We'll see that $\det(A - \lambda I)$ is always a polynomial in λ, and it's called the characteristic polynomial for A.

The eigenvalues are the solutions of $\boxed{\det(A - \lambda I) = 0}$, and this is called the characteristic equation for A.

Example

Find the eigenvalues of $\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$.

Since $A-\lambda I = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix}$,

the characteritic equation is

$$
\begin{aligned}
0 &= \det(A-\lambda I) \\
&= (5-\lambda)(5-\lambda) - 3^2 \\
&= \lambda^2 - 10\lambda + 16 \\
&= (\lambda-2)(\lambda-8).
\end{aligned}
$$

Thus the eigenvalues of A are

2 & 8.

Example

Find the characteristic equation of

$$A = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Recall we can compute 3×3 determinants by cofactor expansion.

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{pmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} - 3 \begin{vmatrix} 3 & 2 \\ 1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 3 & -\lambda \\ 1 & 2 \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 4) - 3(-3\lambda - 2) + (6 + \lambda)$$

$$= -\lambda^3 + \lambda(4 + 9 + 1) + 6 + 6$$

$$= -\lambda^3 + 14\lambda + 12 = 0.$$

The eigenvalues are the solutions of this characteristic equation.

Example

Find the characteric equation of

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -1 & 4 & 2 & 0 & 0 \\ 8 & 6 & -3 & 0 & 0 \\ 5 & -2 & 4 & -1 & 1 \end{pmatrix}$$

Determinants of triangular matrices are easy!

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 & 0 & 0 \\ 2 & 1-\lambda & 0 & 0 & 0 \\ -1 & 4 & 2-\lambda & 0 & 0 \\ 8 & 6 & -3 & -\lambda & 0 \\ 5 & -2 & 4 & -1 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(1-\lambda)(2-\lambda)(-\lambda)(1-\lambda)$$

Thus A has eigenvalues $0, 1, 2$ and 3.

The eigenvalue 1 has multiplicity 2, because the factor $(1-\lambda)$ appears twice in the characteristic polynomial.

The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic equation.

Dynamical systems

A dynamical system is a system described by a difference equation $x_{k+1} = Ax_k$ for some fixed matrix A.

(We've already studied the special case of finite Markov chains were the matrix A was a stochastic matrix, i.e. all columns summed to 1, and all entries positive)

Strategy:
- Find a basis $\mathcal{B}$ of eigenvectors for A, if possible.
- Express the initial conditions in this basis:

$$x_0 = \sum c_i b_i$$

- We can now easily predict the future!

$$x_n = A^n x_0 = \sum c_i A^n b_i = \sum c_i \lambda_i^n b_i$$

Example In 2015, 800,000 people live in a certain city, and 500,000 people live in its suburbs.

If each year 7% of the city residents move to the suburbs, and 3% of the suburbs' residents move to the city,

how many live in each in any given future year?

We can describe this system as

$$x_0 = \begin{pmatrix} 8 \\ 5 \end{pmatrix}, \quad x_{k+1} = M x_k, \quad M = \begin{pmatrix} 0.93 & 0.03 \\ 0.07 & 0.97 \end{pmatrix}$$

notice this is actually a Markov chain!

Let's begin by finding the eigenvalues!

Then $E_1 = \text{Nul}(M - I) = \text{Nul}\begin{pmatrix} -0.07 & 0.03 \\ 0.07 & -0.03 \end{pmatrix} = \text{Nul}\begin{pmatrix} 7 & -3 \\ 0 & 0 \end{pmatrix}$

which is spanned by the eigenvector $v_1 = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$.

Next $E_{0.9} = \text{Nul}(M - 0.9I) = \text{Nul}\begin{pmatrix} 0.03 & 0.03 \\ 0.07 & 0.07 \end{pmatrix} = \text{Nul}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

which is spanned by the eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Write the initial state in the basis of eigenvectors!

$$x_0 = \begin{pmatrix} 8 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 7 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 1 & 8 \\ 7 & -1 & 5 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 1.3 \\ 0 & 1 & 4.1 \end{array}\right)$$

so $c_1 = 1.3, \quad c_2 = 4.1$

$$x_0 = 1.3 v_1 + 4.1 v_2.$$

Now it's easy to see the long term behaviour:

$$x_k = M^k x_0 = (1.3) M^k v_1 + (4.1) M^k v_2$$

$$= (1.3) v_1 + (4.1)(0.9)^k v_2$$

$$= \begin{pmatrix} 3.9 \\ 9.1 \end{pmatrix} + (0.9)^k \begin{pmatrix} 4.1 \\ -4.1 \end{pmatrix}$$

For example, in ~~2050, $k=50$, $(0.9)^{50} \approx$~~

2035, $k=20$, $(0.9)^{20} \approx 0.12$

so $x_{10} \approx \begin{pmatrix} 3.9 \\ 9.1 \end{pmatrix} + \begin{pmatrix} 0.50 \\ -0.50 \end{pmatrix} = \begin{pmatrix} 4.4 \\ 8.6 \end{pmatrix}$

In the long term, as $k \to \infty$, $(0.9)^k \to 0$, so

$x_k \to \begin{pmatrix} 3.9 \\ 9.1 \end{pmatrix}$, and we expect 390,000 people living in the city and 910,000 living in the suburbs.