Overview

The previous lecture introduced eigenvalues and eigenvectors. We'll review these definitions before considering the following question:

Question

Given a square matrix A, how can you find the eigenvalues of A?

We'll discuss an important tool for answering this question: the characteristic equation.

Lay, §5.2

Eigenvalues and eigenvectors

Definition

An *eigenvector* of an $n \times n$ matrix A is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is an *eigenvalue* for A.

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Given any matrix, we can study the associated linear transformation. One way to understand this function is by identifying the set of vectors for which the transformation is just scalar multiplication.

Example

Example 1

Let $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$.

Then $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for the eigenvalue 2:

$$A\mathbf{u} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\mathbf{u}.$$

Also, $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector for the eigenvalue -1:

$$A\mathbf{v} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = -\mathbf{v}.$$

Finding Eigenvalues

Suppose we know that $\lambda \in \mathbb{R}$ is an eigenvalue for A. That is, for some $\mathbf{x} \neq \mathbf{0}$,

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Then we solve for an eigenvector $\mathbf{x}$ by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

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$$\begin{gathered}
\mathbf{x} \text{ must be non zero} \\
\Downarrow \\
(A - \lambda I)\mathbf{x} = \mathbf{0} \text{ must have non trivial solutions} \\
\Downarrow \\
(A - \lambda I) \text{ is not invertible} \\
\Downarrow \\
\det(A - \lambda I) = 0.
\end{gathered}$$

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Solve $\det(A - \lambda I) = 0$ for λ to find the eigenvalues of the matrix A.

The eigenvalues of a square matrix A are the solutions of the characteristic equation.

$$\textit{the characteristic polynomial}: \det(A - \lambda I)$$

$$\textit{the characteristic equation}: \det(A - \lambda I) = \mathbf{0}$$

Examples

Example 2

Consider the matrix

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

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Since

$$A - \lambda I = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5 - \lambda & 3 \\ 3 & 5 - \lambda \end{bmatrix},$$

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$$A - \lambda I = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix},$$

The equation $\det(A - \lambda I) = \mathbf{0}$ becomes

$$\begin{aligned} (5-\lambda)(5-\lambda) - 9 &= 0 \\ \lambda^2 - 10\lambda + 16 &= 0 \\ (\lambda - 8)(\lambda - 2) &= 0 \\ \Rightarrow \lambda = 2, \lambda = 8. \end{aligned}$$

Example 3

Find the characteristic equation for the matrix

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For a 3×3 matrix, recall that a determinant can be computed by cofactor expansion.

$$A - \lambda I = \begin{bmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{bmatrix}$$

$$
\begin{aligned}
\det(A-\lambda I) &= \det\begin{bmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{bmatrix} \\
&= -\lambda\begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 3 & -\lambda \\ 1 & 2 \end{vmatrix} \\
&= -\lambda(\lambda^2-4) - 3(-3\lambda-2) + (6+\lambda) \\
&= -\lambda^3 + 4\lambda + 9\lambda + 6 + 6 + \lambda \\
&= -\lambda^3 + 14\lambda + 12
\end{aligned}
$$

Hence the characteristic equation is

$$-\lambda^3 + 14\lambda + 12 = 0.$$

The eigenvalues of A are the solutions to the characteristic equation.

Example 4

Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -1 & 4 & 2 & 0 & 0 \\ 8 & 6 & -3 & 0 & 0 \\ 5 & -2 & 4 & -1 & 1 \end{bmatrix}$$

Find the characteristic equation for this matrix.

Observe that

$$
\begin{aligned}
\det(A - \lambda I) &= \begin{bmatrix} 3-\lambda & 0 & 0 & 0 & 0 \\ 2 & 1-\lambda & 0 & 0 & 0 \\ -1 & 4 & 2-\lambda & 0 & 0 \\ 8 & 6 & -3 & -\lambda & 0 \\ 5 & -2 & 4 & -1 & 1-\lambda \end{bmatrix} \\
&= (3-\lambda)(1-\lambda)(2-\lambda)(-\lambda)(1-\lambda) \\
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Thus A has eigenvalues $0, 1, 2$ and 3. The eigenvalue 1 is said to have *multiplicity* 2 because the factor $1-\lambda$ occurs twice in the characteristic polynomial.

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&= (3-\lambda)(1-\lambda)(2-\lambda)(-\lambda)(1-\lambda) \\
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In general the **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Similarity

The next theorem illustrates the use of the characteristic polynomial, and it provides a basis for several iterative methods that *approximate* eigenvalues.

Definition (Similar matrices)

If A and B are $n \times n$ matrices, then A is **similar** to B if there is an invertible matrix P such that

$$P^{-1}AP = B$$

or equivalently,

$$A = PBP^{-1}.$$

We say that A and B **are similar**. Changing A into $P^{-1}AP$ is called a **similarity transformation**.

Theorem

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Proof.
If $B = P^{-1}AP$, then

$$\begin{aligned} B - \lambda I &= P^{-1}AP - \lambda P^{-1}P \\ &= P^{-1}(AP - \lambda P) \\ &= P^{-1}(A - \lambda I)P. \end{aligned}$$

Hence

$$\det(B - \lambda I) = \det\left[P^{-1}(A - \lambda I)P\right]$$

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Application to dynamical systems

A dynamical system is a system described by a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$. Such an equation was used to model population movement in Lay 1.10 and it is the sort of equation used to model a Markov chain. Eigenvalues and eigenvectors provide a key to understanding the evolution of a dynamical system.

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1. If you can, find a basis $\mathcal{B}$ of eigenvectors:

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$$\mathbf{x_0} = c_1\mathbf{b_1} + c_2\mathbf{b_2}.$$
3. Since A multiplies each eigenvector by the corresponding eigenvalue, this makes it easy to see what happens after many iterations:
$$A^n\mathbf{x_0} = A^n(c_1\mathbf{b_1} + c_2\mathbf{b_2}) = c_1A^n\mathbf{b_1} + c_2A^n\mathbf{b_2} = c_1\lambda_1^n\mathbf{b_1} + c_2\lambda_2^n\mathbf{b_2}.$$

Examples

Example 5

In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 3% of the suburban population moves to the city. In 2000 there were 800,000 residents in the city and 500,000 residents in the suburbs. We want to investigate the result of this migration in the long term.

The migration matrix M is given by

$$M = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}.$$

- The first step is to find the eigenvalues of M.

The characteristic equation is given by

$$
\begin{aligned}
0 &= \det \begin{bmatrix} .93-\lambda & .03 \\ .07 & .97-\lambda \end{bmatrix} \\
&= (.93-\lambda)(.97-\lambda) - (.03)(.07) \\
&= \lambda^2 - 1.9\lambda + .9021 - .0021 \\
&= \lambda^2 - 1.9\lambda + .9000 \\
&= (\lambda - 1)(\lambda - .9)
\end{aligned}
$$

So the eigenvalues are $\lambda = 1$ and $\lambda = 0.9$.

$$E_1 = \mathrm{Nul} \begin{bmatrix} -.07 & .03 \\ .07 & -.03 \end{bmatrix} = \mathrm{Nul} \begin{bmatrix} 7 & -3 \\ 0 & 0 \end{bmatrix}$$

This gives an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

$$E_{.9} = \text{Nul} \begin{bmatrix} .03 & .03 \\ .07 & .07 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and an eigenvector for this space is given by $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

- The next step is to write $\mathbf{x}_0$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

The initial vector $\mathbf{x}_0$ describes the initial population (in 2000), so writing in 100,000's we will put $\mathbf{x}_0 = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$.

There exist weights c_1 and c_2 such that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{1}$$

To find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ we do the following row reduction:

$$\begin{bmatrix} 3 & 1 & 8 \\ 7 & -1 & 5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1.3 \\ 0 & 1 & 4.1 \end{bmatrix}$$

So

$$\mathbf{x}_0 = 1.3\mathbf{v}_1 + 4.1\mathbf{v}_2. \tag{2}$$

- We can now look at the long term behaviour of the system. Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of M, with $M\mathbf{v}_1 = \mathbf{v}_1$ and $M\mathbf{v}_2 = .9\mathbf{v}_2$, we can compute each $\mathbf{x}_k$:

$$\begin{aligned}
\mathbf{x}_1 = M\mathbf{x}_0 &= c_1 M\mathbf{v}_1 + c_2 M\mathbf{v}_2 \\
&= c_1\mathbf{v}_1 + c_2(0.9)\mathbf{v}_2 \\
\mathbf{x}_2 = M\mathbf{x}_1 &= c_1 M\mathbf{v}_1 + c_2(0.9)M\mathbf{v}_2 \\
&= c_1\mathbf{v}_1 + c_2(0.9)^2\mathbf{v}_2
\end{aligned}$$

In general we have

$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(0.9)^k\mathbf{v}_2, \quad k = 0, 1, 2, \dots,$$

that is

$$\mathbf{x}_k = 1.3\begin{bmatrix} 3 \\ 7 \end{bmatrix} + 4.1(0.9)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; k = 0, 1, 2, \dots$$

As $k \to \infty$, $(0.9)^k \to 0$, and $\mathbf{x}_k \to 1.3\mathbf{v}_1$, which is $\begin{bmatrix} 3.9 \\ 9.1 \end{bmatrix}$. This indicates that in the long term 390,000 are expected to live in the city, while 910,000 are expected to live in the suburbs.

Example 6

Let $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$. We analyse the long-term behaviour of the dynamical system defined by $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $(k = 0, 1, 2, \ldots)$, with $\mathbf{x}_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$.

- As in the previous example we find the eigenvalues and eigenvectors of the matrix A.

$$\begin{aligned} 0 &= \det \begin{bmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{bmatrix} \\ &= (0.8-\lambda)(0.9-\lambda) - (0.1)(0.2) \\ &= \lambda^2 - 1.7\lambda + 0.7 \\ &= (\lambda - 1)(\lambda - 0.7) \end{aligned}$$

So the eigenvalues are $\lambda = 1$ and $\lambda = 0.7$. Eigenvalues corresponding to these eigenvalues are multiples of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

respectively. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is clearly a basis for $\mathbb{R}^2$.

- The next step is to write $\mathbf{x}_0$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

There exist weights c_1 and c_2 such that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{3}$$

To find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ we do the following row reduction:

$$\begin{bmatrix} 1 & 1 & 0.7 \\ 2 & -1 & 0.3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0.333 \\ 0 & 1 & 0.367 \end{bmatrix}$$

So

$$\mathbf{x}_0 = 0.333\mathbf{v}_1 + 0.367\mathbf{v}_2. \tag{4}$$

- We can now look at the long term behaviour of the system.

As in the previous example, since $\lambda_1 = 1$ and $\lambda_2 = 0.7$ we have

$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(0.7)^k\mathbf{v}_2, \quad k = 0, 1, 2, \ldots,$$

This gives

$$\mathbf{x}_k = 0.333\begin{bmatrix}1\\2\end{bmatrix} + 0.367(0.7)^k\begin{bmatrix}1\\-1\end{bmatrix},\ k = 0, 1, 2, \ldots$$

As $k \to \infty$, $(0.7)^k \to 0$, and $\mathbf{x}_k \to 0.333\mathbf{v}_1$, which is $\begin{bmatrix}1/3\\2/3\end{bmatrix}$. This is the steady state vector of the Markov chain described by A.

Some Numerical Notes

- Computer software such as Mathematica and Maple can use symbolic calculation to find the characteristic polynomial of a moderate sized matrix. There is no formula or finite algorithm to solve the characteristic equation of a general $n \times n$ matrix for $n \geq 5$.
- The best numerical methods for finding eigenvalues avoid the characteristic equation entirely. Several common algorithms for estimating eigenvalues are based on the Theorem on Similar matrices. Another technique, called *Jacobi's method* works when $A = A^T$ and computes a sequence of matrices of the form

$$A_1 = A \text{ and } A_{k+1} = P_k^{-1} A_k P_k, \; k = 1, 2, \ldots .$$

Each matrix in the sequence is similar to A and has the same eigenvalues as A. The non diagonal entries of A_{k+1} tend to 0 as k increases, and the diagonal entries tend to approach the eigenvalues of A.