Sample Question: Lines & Planes

Let P be the plane in $\mathbb{R}^3$ defined by the equation $2x + y - z = 1$, and let L be the line through the point $(1, 1, 1)$ which is orthogonal to P.

1. Find an equation for P of the form $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ for some vector $\mathbf{n}$ and some vector $\mathbf{r_0}$.
2. Find an equation for L.
3. Let Q be the plane containing L and the point $(1, 1, 2)$. Find an equation for Q.

Sample Question: Bases & Coordinates

The set $\mathcal{B} = \{t+1, 1+t^2, 3-t^2\}$ is a basis for $\mathbb{P}_2$.

1. If $p(t) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{B}}$, express p in the form $p(t) = a + bt + ct^2$.
2. Find the coordinate vector of the polynomial $q(t) = 2 - 2t$ with respect to $\mathcal{B}$ coordinates.

Sample Question: Vector Spaces

Decide whether each of the following sets is a vector space. If it is a vector space, state its dimension. If it is not a vector space, explain why.

1. A is the set of 2×2 matrices whose entries are integers.
2. B is the set of vectors in $\mathbb{R}^3$ which are orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.
3. C is the set of polynomials whose derivative is 0:

$$C = \{p(x) \in \mathbb{P} \mid \frac{d}{dx} p(x) = 0\}.$$

Sample Question: Linear transformations

A linear transformation $T : M_{2\times 2} \to M_{2\times 2}$ is defined by:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

(a) Calculate $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.

(b) Which, if any, of the following matrices are in $\ker(T)$?

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

(c) Which, if any, of the following matrices are in $\text{range}(T)$?

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Find the kernel of T and explain why T is not one to one.

(e) Explain why T does not map $M_{2\times 2}$ onto $M_{2\times 2}$.

Sample Question: Subspaces associated to a matrix

Consider the matrix A:

$$\begin{bmatrix} 2 & -4 & 0 & 2 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 1 & 4 \end{bmatrix}.$$

(i) Find a basis for $\operatorname{Nul} A$.

(ii) Find a basis for $\operatorname{Col} A$.

(iii) Consider the linear transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$. Give a geometric description of the range of T_A as a subspace of $\mathbb{R}^3$. What is its dimension? Does it pass through the origin?