Geometry of $\mathbb{R}^3$

- What information determines a line or a plane in $\mathbb{R}^3$?
- Can you convert between the different types of equations defining a line or plane?
- How do you check two lines are parallel/orthogonal.
- How do you compute distances between
  i) a point and a line ii) a point and a plane iii) two lines.
- How do you calculate the angle between two vectors.
- What do scalar and vector projections mean? What are the formulas?

Vector spaces

- What is a vector space? Give some examples!
- What is a subspace? How do you check if a subset of a vector space is a subspace.
- What is a linear transformation? Give some examples!
  - What does one-to-one mean? Onto?
- What does it mean for a set of vectors to be linearly independent?

  How do you check this?
-
  - what is a spanning set?
  - what is span S, if S is a set of vectors.

Bases for vector spaces

- What is a basis?
- If the dimension of V is n, then V is isomorphic to $\mathbb{R}^n$.
  What does this mean? Why is true?
- What are the coordinates of a vector w.r.t. a basis?
- How do you compute the change of coordinates matrix?

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \begin{pmatrix} | & | & & | \\ [b_1]_{\mathcal{E}} & [b_2]_{\mathcal{E}} & \cdots & [b_n]_{\mathcal{E}} \\ | & | & & | \end{pmatrix}$$

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} [x]_{\mathcal{B}} = [x]_{\mathcal{E}}.$$

- in a n-dimensional vector space:
  - any n linearly independent vectors form a basis
  - any n vectors which span, form a basis
  - any set of vectors which span V, contain some subset which is a basis.
  - any set of linearly independent vectors in V can be extended to a basis of V.

- How do you find a basis for the nullspace of a matrix?
  ~~&~~ Column space? Row space?
  The kernel of the associated linear transformation
  $T_A : \mathbb{R}^n \to \mathbb{R}^m \qquad T_A(x) = Ax$?

Sample Question: Lines & Planes

Let P be the plane in $\mathbb{R}^3$ defined by the equation $2x + y - z = 1$, and let L be the line through the point $(1, 1, 1)$ which is orthogonal to P.

1. Find an equation for P of the form $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ for some vector $\mathbf{n}$ and some vector $\mathbf{r_0}$.
2. Find an equation for L.
3. Let Q be the plane containing L and the point $(1, 1, 2)$. Find an equation for Q.

(1) n should be ~~the~~ a normal vector to the plane, and r_0 should be any vector contained in the plane.

We can find an r_0 by substituting in $x=0$, $y=0$, and finding $z=-1$. $r_0 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

~~Pla~~ A plane written as $Ax + By + Cz + D = 0$ has $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ as a normal vector.

Thus we can use $n = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

So P is defined by $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \left(r - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right) = 0$.

(2) L is defined by $r = r_0 + tv$

where $r_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and

v is the direction vector, i.e. a normal to P,

$$v = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

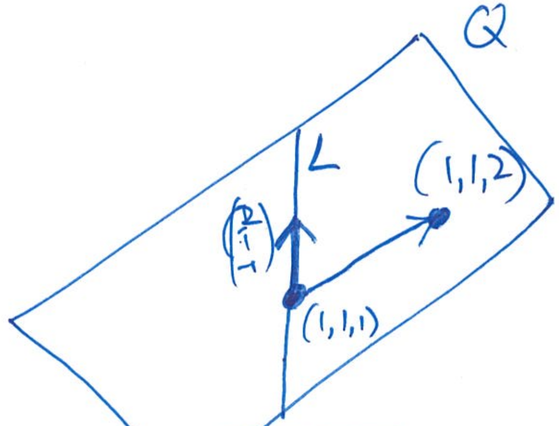
(3) Let's write Q as the solutions to $\boxed{n \cdot (r - r_0) = 0}$

where n is a normal vector to Q, and $\boxed{r_0 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}$ is a point on Q.

To find a normal vector, we can compute the cross product of two independent vectors that are parallel to Q.

One is the direction vector for L, $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$,

We can take for the second, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ the vector between the two points on Q that we know.



$$n = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = i \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + k \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = \boxed{i - 2j} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

Sample Question: Bases & Coordinates

The set $\mathcal{B} = \{t+1, 1+t^2, 3-t^2\}$ is a basis for $\mathbb{P}_2$.

1. If $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, express p in the form $p(t) = a + bt + ct^2$.
2. Find the coordinate vector of the polynomial $q(t) = 2 - 2t$ with respect to $\mathcal{B}$ coordinates.

① If $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, that means

$$p(t) = \underline{1} \cdot (t+1) + \underline{1} \cdot (1+t^2) + -1 \cdot (3-t^2)$$

$$= -1 + t + 2t^2.$$

② Find $[2-2t]_{\mathcal{B}}$. We need to solve

$$2-2t = a(t+1) + b(1+t^2) + c(3-t^2)$$

$$\left.\begin{aligned} 2 &= a+b+3c \\ -2 &= a \\ 0 &= b-c \end{aligned}\right\} \Rightarrow \begin{aligned} &a=-2, \quad b=c \\ &2=-2+4c \\ &c=1. \end{aligned}$$

$$[2-2t]_{\mathcal{B}} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Check!

Let $\mathcal{E} = \{1, t, t^2\}$

$$[2-2t]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} [2-2t]_{\mathcal{E}}$$

$$= \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$= \underset{\mathcal{E} \leftarrow \mathcal{B}}{P^{-1}} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

Check this comes out the same!

Sample Question: Vector Spaces

Decide whether each of the following sets is a vector space. If it is a vector space, state its dimension. If it is not a vector space, explain why.

1. A is the set of 2×2 matrices whose entries are integers.
2. B is the set of vectors in $\mathbb{R}^3$ which are orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.
3. C is the set of polynomials whose derivative is 0:

$$C = \{p(x) \in \mathbb{P} \mid \frac{d}{dx} p(x) = 0\}.$$

① The set of 2×2 matrices with integer entries is a subset of $M_{2\times 2}$.

We need to check.

- Does A contain 0? Yes.
- Is A closed under vector addition? Yes.
- Is A closed under scalar multiplication?

$$\frac{1}{2}\cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\in A} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \notin A.$$

It's not a vector space!

② $B = \mathrm{Nul}\cancel{\begin{bmatrix}1\\0\\2\end{bmatrix}}\begin{bmatrix}1 & 0 & 2\end{bmatrix}$, and nullspaces are always vector spaces.

(of course, we could check the axioms directly).

What $\dim B$?

We can use the rank-nullity theorem.

$$\mathrm{rank}\cancel{\begin{bmatrix}1\\0\\2\end{bmatrix}}\begin{bmatrix}1 & 0 & 2\end{bmatrix} = 1.$$

$$\mathrm{rank}\cancel{\begin{bmatrix}1\\0\\2\end{bmatrix}} + \dim \mathrm{Nul}\cancel{\begin{bmatrix}1\\0\\2\end{bmatrix}} = \cancel{\#\text{ of rows}}\ \#\text{ of columns} = 3$$

$\dim B = 2.$

$$B = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = 0\right\}$$

$$= \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{pmatrix}1 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 0\right\}$$

Sample Question: Linear transformations

A linear transformation $T : M_{2\times 2} \to M_{2\times 2}$ is defined by:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

(a) Calculate $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.

(b) Which, if any, of the following matrices are in $\ker(T)$?

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

(c) Which, if any, of the following matrices are in $\mathrm{range}(T)$?

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Find the kernel of T and explain why T is not one to one.

(e) Explain why T does not map $M_{2\times 2}$ onto $M_{2\times 2}$.

a) $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} a-b & -a+b \\ c-d & -c+d \end{pmatrix}$.

b) Only $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

c) $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \in$ Range T because

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = T\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$ ....

d) $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underset{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}{\underset{\shortparallel}{0}} \Longleftrightarrow a=b$ and $c=d$.

$\ker T = \left\{ \begin{pmatrix} a & a \\ c & c \end{pmatrix} \right\}$ which has a basis $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$

T is not one-to-one because it has a non-trivial kernel.

e) Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin$ Range T, T is not onto.

Sample Question: Subspaces associated to a matrix

Consider the matrix A:

$$\begin{bmatrix} 2 & -4 & 0 & 2 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 1 & 4 \end{bmatrix}.$$

(i) Find a basis for $\operatorname{Nul} A$.

(ii) Find a basis for $\operatorname{Col} A$.

(iii) Consider the linear transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$. Give a geometric description of the range of T_A as a subspace of $\mathbb{R}^3$. What is its dimension? Does it pass through the origin?