Computing eigenvalues in practice

The characteristic polynomial of an $n \times n$ matrix A will have degree n. You can try to find it by row-reducing $A - \lambda I$.

But if n is too large (>5) you can't solve the characteristic equation exactly.

Typically good numerical algorithms avoid computing the characteristic polynomial!

A famous example is the Jacobi algorithm for approximating the eigenvalues of a symmetric matrix $A = A^T$.

Def^n If A and B are $n \times n$ matrices, we say

A is similar to B

if there is an invertible matrix P such that

$$P^{-1}AP = B$$

or equivalently

$$A = PBP^{-1}.$$

Changing A into $P^{-1}AP$ is called a similarity transformation.

Theorem If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues.

Proof: If $B = P^{-1}AP$, then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P$$
$$= P^{-1}(A - \lambda I)P$$

so

$$\det(B - \lambda I) = \det(P^{-1}(A - \lambda I)P)$$
$$= \det(P^{-1})\det(A - \lambda I)\det(P)$$
$$= \det(P^{-1})\det(P)\det(A - \lambda I)$$
$$= \det(P^{-1}P)\det(A - \lambda I)$$
$$= \det(I)\det(A - \lambda I)$$
$$= \det(A - \lambda I).$$

Warning: the converse is not true!

Example the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ have the same characteristic polynomial:

$$\det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda) = 1-2\lambda+\lambda^2$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda) = 1-2\lambda+\lambda^2.$$

But they are not similar. For any invertible matrix P

$$P^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}P = P^{-1}P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This is the first place we see a difference between the algebraic multiplicity

(the number of times λ appears as a root of the characteristic polynomial)

and the geometric multiplicity

(the dimension of the λ-eigenspace)

of an eigenvalue.

For $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $E_1 = \mathrm{Nul}\begin{pmatrix}0&0\\0&0\end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$, so the algebraic and geometric multiplicities are both 2.

For $\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $E_1 = \mathrm{Nul}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$, so the algebraic multiplicity is 1, but the geometric multiplicity is only 1.

<u>Recall</u>: matrices A and B are similar if there exists an invertible matrix P so that

$$A = P B P^{-1}$$

(necessarily, A and B are square and the same size)

This is an <u>equivalence relation</u>:

i) any square matrix A is similar to itself

ii) if A is similar to B, then B is similar to A

iii) if A is similar to B, and B is similar to C, then A is similar to C

(if $A = P B P^{-1}$, $B = Q C Q^{-1}$ then

$$A = P Q C Q^{-1} P^{-1} = (QP) C (QP)^{-1}.)$$

What is the 'simplest' matrix in the equivalence class of a given matrix?

Often it will be a <u>diagonal matrix</u>.

We say A is **diagonalisable** if it is similar to some diagonal matrix D.

(That is, there exists an invertible matrix P so

$$A = PDP^{-1}.)$$

What's so great about diagonal matrices?

It's easy to answer all these questions:

- What are the eigenvalues of D?
  What is the dimension of each eigenspace?
  What are the basis vectors for these eigenspaces?
- What is the determinant of D?
- Is D invertible?
- ~~What is the characteristic polynomial of D?~~
- What is D^k, for $k = 0, 1, 2, 3, \dots$

Example Can you answer all these questions for

$$D = \begin{pmatrix} 10^{50} & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & -2.7 \end{pmatrix}?$$

Example

Let $A=\begin{pmatrix}1 & 3\\ 2 & 2\end{pmatrix}$.

$$A=PDP^{-1},$$

where

$$P=\begin{pmatrix}1 & 3\\ 1 & -2\end{pmatrix}$$

and $D=\begin{pmatrix}4 & 0\\ 0 & -1\end{pmatrix}$.

Find a explicit formula for A^k.

$$A=PDP^{-1}$$

$$A^2=PD\underbrace{\cancel{P^{-1}P}}_{I}DP^{-1}$$

$$=PD^2P^{-1}$$

$$A^3=PD^2\underbrace{\cancel{P^{-1}P}}_{I}DP^{-1}$$

$$=PD^3P^{-1}$$

and so on.

$$A^k=PD^kP^{-1}$$

$$=\begin{pmatrix}1 & 3\\ 1 & -2\end{pmatrix}\begin{pmatrix}4^k & 0\\ 0 & (-1)^k\end{pmatrix}\begin{pmatrix}\frac{2}{5} & \frac{3}{5}\\ \frac{1}{5} & -\frac{1}{5}\end{pmatrix}$$

$$=\begin{pmatrix}4^k & 3(-1)^k\\ 4^k & -2(-1)^k\end{pmatrix}\begin{pmatrix}\frac{2}{5} & \frac{3}{5}\\ \frac{1}{5} & -\frac{1}{5}\end{pmatrix}=\begin{pmatrix}\frac{2}{5}4^k+\frac{3}{5}(-1)^k & \frac{3}{5}4^k-\frac{3}{5}(-1)^k\\ \frac{2}{5}4^k-\frac{2}{5}(-1)^k & \frac{3}{5}4^k+\frac{2}{5}(-1)^k\end{pmatrix}.$$

How can we tell if a matrix is diagonalisable?

In that example, notice

$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1 & 3\\2 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 4\begin{pmatrix}1\\1\end{pmatrix}$$

$$A\begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}1 & 3\\2 & 2\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = -1\begin{pmatrix}3\\-2\end{pmatrix}.$$

The columns of P were eigenvectors of A, and the eigenvector was the corresponding diagonal entry in D.

In general, if $AP = PD$,

$$A\begin{bmatrix} | & | & & | \\ p_1 & p_2 & \cdots & p_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ap_1 & Ap_2 & \cdots & Ap_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ p_1 & p_2 & \cdots & p_n \\ | & | & & | \end{bmatrix} D$$

$$= \begin{bmatrix} | & | & & | \\ d_1 p_1 & d_2 p_2 & \cdots & d_n p_n \\ | & | & & | \end{bmatrix}$$

so the columns of P are eigenvectors.

P is invertible exactly if they are all independent.

Conversely, if A has n independent eigenvectors,

let $P = \begin{bmatrix} | & | & & | \\ p_1 & p_2 & \cdots & p_n \\ | & | & & | \end{bmatrix}$, then $AP = PD$, so $A = PDP^{-1}$,

where D is the diagonal matrix with the eigenvalues on the diagonal.