## Theorem (The Diagonalisation Theorem)

Let A be an  $n \times n$  matrix. Then A is diagonalisable if and only if A has n linearly independent eigenvectors.

 $P^{-1}AP$  is a diagonal matrix D if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors of A in the same order.

## Example 1

Find a matrix P that diagonalises the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

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• The characteristic polynomial is given by

$$det(A - \lambda I) = det \begin{bmatrix} -1 - \lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1 - \lambda \end{bmatrix}.$$
$$= (-1 - \lambda)(-\lambda)(-1 - \lambda) + \lambda$$
$$= -\lambda^2(\lambda + 2).$$

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The eigenvalues of A are  $\lambda = 0$  (of multiplicity 2) and  $\lambda = -2$  (of multiplicity 1).

• The eigenspace  $E_0$  has a basis consisting of the vectors

$$\mathbf{p}_1 = \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1\\0\\1 \end{bmatrix}$$

and the eigenspace  $E_{\!-2}$  has a basis consisting of the vector

$$\mathbf{p}_3 = \begin{bmatrix} -1\\ 3\\ 1 \end{bmatrix}$$

It is easy to check that these vectors are linearly independent.

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So if we take

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

then P is invertible.

It is easy to check that AP = PD where  $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$  $AP = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}$  $PD = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}.$ 

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## Example 2

Can you find a matrix P that diagonalises the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}?$$

• The characteristic polynomial is given by

$$det(A - \lambda I) = det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4 - \lambda \end{bmatrix}$$
  
=  $(-\lambda) [-\lambda(4 - \lambda) + 5] - 1(-2)$   
=  $-\lambda^3 + 4\lambda^2 - 5\lambda + 2$   
=  $-(\lambda - 1)^2(\lambda - 2)$ 

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This means that A has eigenvalues  $\lambda = 1$  (of multiplicity 2) and  $\lambda = 2$  (of multiplicity 1).

• The corresponding eigenspaces are

$$E_1 = \text{Span} \left\{ \begin{bmatrix} 1\\1\\1 \end{bmatrix} \right\}, E_2 = \text{Span} \left\{ \begin{bmatrix} 1\\2\\4 \end{bmatrix} \right\}.$$

Note that although  $\lambda = 1$  has multiplicity 2, the corresponding eigenspace has dimension 1. This means that we can only find 2 linearly independent eigenvectors, and by the Diagonalisation Theorem A is not diagonalisable.

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Example 3Consider the matrix $ A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 0 & 1 \end{bmatrix} $ Why is A diagonalisable?Since A is upper triangular, it's easy to see that it has three distinct eigenvalues: $\lambda_1 = 2$ , $\lambda_2 = 5$ and $\lambda_3 = 1$ . Eigenvectors corresponding to distinct eigenvalues are has three linearly independent, so A has three linearly independent is of A has three linearly independent, so A has three linearly independent, so A has three linearly independent, so A has the independent is of A to 0 independe		
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diagonalisable? The eigenvalues are $\lambda = 4$ with multiplicity 2, and $\lambda = 2$ with multiplicity 2. A/Prof Scott Morrison (ANU) MATHONE Second Second 2 (A)	$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$	
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The eigenspace 
$$E_2$$
 is given by  

$$E_2 = \operatorname{Nul} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{Span} \left\{ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$
and has dimension 2.  

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$$P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

$$\frac{\operatorname{Afted seek Merroe (MU}} = \operatorname{Mark to the eigenspace for } \lambda_k \text{ is less than or equal to its multiplicity.}$$

$$\frac{\operatorname{Afted seek Merroe (MU}}{1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

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