Theorem (The Diagonalisation Theorem)

Let A be an $n \times n$ matrix. Then A is diagonalisable if and only if A has n linearly independent eigenvectors.

$P^{-1}AP$ is a diagonal matrix D if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors of A in the same order.

Example 1

Find a matrix P that diagonalises the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

- The characteristic polynomial is given by

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{bmatrix}. \\ &= (-1-\lambda)(-\lambda)(-1-\lambda) + \lambda \\ &= -\lambda^2(\lambda + 2). \end{aligned}$$

The eigenvalues of A are $\lambda = 0$ (of multiplicity 2) and $\lambda = -2$ (of multiplicity 1).

- The eigenspace E_0 has a basis consisting of the vectors

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

  and the eigenspace E_{-2} has a basis consisting of the vector

$$\mathbf{p}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

  It is easy to check that these vectors are linearly independent.

- So if we take

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

then P is invertible.

It is easy to check that $AP = PD$ where $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$$AP = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}.$$

Example 2

Can you find a matrix P that diagonalises the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}?$$

- The characteristic polynomial is given by

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{bmatrix} \\ &= (-\lambda)\left[-\lambda(4-\lambda) + 5\right] - 1(-2) \\ &= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 \\ &= -(\lambda - 1)^2(\lambda - 2) \end{aligned}$$

This means that A has eigenvalues $\lambda = 1$ (of multiplicity 2) and $\lambda = 2$ (of multiplicity 1).

- The corresponding eigenspaces are

$$E_1 = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}, E_2 = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\right\}.$$

Note that although $\lambda = 1$ has multiplicity 2, the corresponding eigenspace has dimension 1. This means that we can only find 2 linearly independent eigenvectors, and by the Diagonalisation Theorem A is not diagonalisable.

Example 3

Consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Why is A diagonalisable?

Since A is upper triangular, it's easy to see that it has three distinct eigenvalues: $\lambda_1 = 2, \lambda_2 = 5$ and $\lambda_3 = 1$. Eigenvectors corresponding to distinct eigenvalues are linearly independent, so A has three linearly independent eigenvectors and is therefore diagonalisable.

Theorem

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalisable.

Example 4

Is the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

diagonalisable?

The eigenvalues are $\lambda = 4$ with multiplicity 2, and $\lambda = 2$ with multiplicity 2.

The eigenspace E_4 is found as follows:

$$\begin{aligned} E_4 &= \text{Nul} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \\ &= \text{Span} \left\{ \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \end{aligned}$$

and has dimension 2.

The eigenspace E_2 is given by

$$\begin{aligned} E_2 &= \text{Nul}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \text{Span}\left\{ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \end{aligned}$$

and has dimension 2.

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is linearly independent.}$$

This implies that $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$ is invertible and $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$.

1. *For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to its multiplicity.*
2. *The matrix A is diagonalisable if and only if the sum of the dimensions of the distinct eigenspaces equals n.*
3. *If A is diagonalisable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.*
4. *If $P^{-1}AP = D$ for a diagonal matrix D, then P is the change of basis matrix from eigenvector coordinates to standard coordinates.*