Theorem Let A be an $n \times n$ matrix.

A is diagonalisable if and only if A has n linearly independent eigenvectors.

In that case, $A = PDP^{-1}$, where P is a matrix whose columns are a basis of eigenvectors, and D is the diagonal matrix whose diagonal entries are the corresponding eigenvalues.

Example

Diagonalise

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$$

The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & 3 \\ 1 & 0 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(-\lambda)(-1-\lambda) + \lambda$$

$$= -\lambda^3 - 2\lambda^2 = -\lambda^2(\lambda+2)$$

So the eigenvalues of A are

$\lambda = 0$ (with algebraic multiplicity 2)

and $\lambda = -2$ (with algebraic multiplicity 1).

Next, let's find bases for the eigenspaces.

$$E_0 = \mathrm{Nul}\begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

$$E_{-2} = \mathrm{Nul}\begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}\right\}.$$

It's easy to see these three vectors are linearly independent.

So, if we take $P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$,

with $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

it is invertible! We should have $A = PDP^{-1}$. Let's check —

$$AP = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{pmatrix}$$

$$PD = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & +2 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{pmatrix}.$$

Example

Why is

$$A=\begin{pmatrix} 2 & -3 & 7 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

diagonalisable?

A is upper triangular, so it's eigenvalues are the diagonal entries:

$$\lambda = 2, \quad \lambda = 5 \quad \text{and} \quad \lambda = 1.$$

Eigenvectors corresponding to distinct eigenvalues are linearly independent

recall last ~~Tuesday~~ Monday!

so A has three linearly independent eigenvectors and therefore is diagonalisable

<u>Thm</u> An $n \times n$ matrix with n distinct eigenvalues is diagonalisable!

Example

Is $A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ diagonalisable?

The eigenvalues 4 and 2 both have multiplicity 2, so we can't use the last theorem.

$$E_4 = \mathrm{Nul}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$E_2 = \mathrm{Nul}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Both have dimension 2, so

$P = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ has independent columns, so is invertible,

and $A = PDP^{-1}$, where $D = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

A is diagonalisable.

Theorem Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$.

1) For $1 \le k \le p$, $\dim E_{\lambda_k} \le$ multiplicity of λ_k as a root of the characteristic polynomial

"geometric multiplicity $\le$ algebraic multiplicity"

2) A is diagonisable $\iff \displaystyle\sum_{k=1}^{p} \dim E_{\lambda_k} = n$

3) If A is diagonalisable, and $\mathcal{B}_k$ is a basis for E_{λ_k}, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_p$ is an eigenvector basis for $\mathbb{R}^n$.

4) If $A = PDP^{-1}$ for a diagonal matrix P, then

$P = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$, the change of basis matrix from eigenvector coordinates to standard coordinates.