Matrices and linear transformations

Recall any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ can be realised as a matrix transformation

$$T(x) = Ax \qquad \text{for all } x \in \mathbb{R}^n,$$

for a unique matrix A.

In fact, this matrix is just

$$A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}.$$

This is called the standard matrix for T.

What about when T is a more general linear transformation, $T: V \to W$?

We can still find an associated matrix, but to do so we have to choose bases for V and for W, and the matrix will depend on these choices.

If $\mathcal{B}$ is a basis for V, and $\mathcal{C}$ is a basis for W, we define

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{T} = \begin{bmatrix} | & | & & | \\ [T(b_1)]_{\mathcal{C}} & [T(b_2)]_{\mathcal{C}} & \cdots & [T(b_n)]_{\mathcal{C}} \\ | & | & & | \end{bmatrix}$$

With this definition, we have the formula

$$\left[T(v)\right]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{T}\left[v\right]_{\mathcal{B}} \qquad \text{for all vectors } v \in V.$$

Notice that our change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from before is a special case of this —

it's just that T is the identity linear transformation

$$T(v) = v$$

from some vector space V to itself,

but we use two different bases $\mathcal{B}$ and $\mathcal{C}$, for before and after:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{T} = \begin{bmatrix} | & & | \\ [T(b_1)]_{\mathcal{C}} & \cdots & [T(b_n)]_{\mathcal{C}} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ [b_1]_{\mathcal{C}} & \cdots & [b_n]_{\mathcal{C}} \\ | & & | \end{bmatrix}.$$

If $T: V \to V$ is a linear transformation from some vector space to itself, and it has a basis of eigenvectors $\mathcal{B} = \{b_1, \dots, b_n\}$, so $Tb_i = \lambda_i b_i$ for some scalars λ_i, then the associated matrix $\underset{\mathcal{B}\leftarrow\mathcal{B}}{T}$ is particularly simple:

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{T} = \begin{bmatrix} | & & | \\ [T(b_1)]_{\mathcal{B}} & \cdots & [T(b_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ [\lambda_1 b_1]_{\mathcal{B}} & \cdots & [\lambda_n b_n]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & 0 \\ 0 & \cdots & & \lambda_n \end{pmatrix}$$

That is, it's just ~~the~~ a diagonal matrix.

In fact, this is just diagonalisation in disguise — if $V = \mathbb{R}^n$,

$$ {}_{\mathcal{E}\leftarrow\mathcal{E}}T = {}_{\mathcal{E}\leftarrow\mathcal{E}}(\mathrm{id}\circ T\circ \mathrm{id}) = {}_{\mathcal{E}\leftarrow\mathcal{B}}\mathrm{id}\ {}_{\mathcal{B}\leftarrow\mathcal{B}}T\ {}_{\mathcal{B}\leftarrow\mathcal{E}}\mathrm{id} $$

$$ = PDP^{-1} $$

where P is the change of basis matrix from ~~$\mathcal{B}$~~
$\mathcal{B}$ (the basis of eigenvectors) to
$\mathcal{E}$ (the standard basis of $\mathbb{R}^n$).

Example

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 3x+y \\ x-y \end{bmatrix}$$

Find the standard matrix for T.

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

So $$\underset{\mathcal{E} \leftarrow \mathcal{E}}{T} = \begin{bmatrix} | & | \\ [T(e_1)]_{\mathcal{E}} & [T(e_2)]_{\mathcal{E}} \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \\ 1 & -1 \end{bmatrix}.$$

Example

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

and $T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$Tx = Ax.$$

~~What is~~

If $B = \{e_1, e_2, -e_1 + e_3\}$,

what is

$${}_{B \leftarrow B}T \; ?$$

Example

Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{D} = \{d_1, d_2\}$ be bases for vector space V and W.

Suppose $T: V \to W$ is defined by

$$T(b_1) = 3d_1 - 5d_2$$

$$T(b_2) = -d_1 + 6d_2$$

$$T(b_3) = 4d_2.$$

What is the matrix $\underset{\mathcal{D} \leftarrow \mathcal{B}}{T}$?

Example

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

and $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x) = Ax$.

If $\mathcal{B} = \{e_1, e_2, -e_1 + e_3\}$ what is $\underset{\mathcal{B} \leftarrow \mathcal{B}}{T}$?

$$T(e_1) = A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2e_1$$

$$[T(e_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

$$T(e_2) = A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -1 \cdot e_2$$

$$[T(e_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

$$T(-e_1 + e_3) = A\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = +1 \cdot (-e_1 + e_3)$$

$$[T(-e_1 + e_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ +1 \end{bmatrix}.$$

$$\boxed{\underset{\mathcal{B} \leftarrow \mathcal{B}}{T} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.}$$

Example

Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by

$$T(p(t)) = \begin{bmatrix} p(0) + p(1) \\ p(-1) \end{bmatrix}.$$

Exercise!

a) Show that T is a linear transformation

b) Find $\underset{\mathcal{E} \leftarrow \mathcal{B}}{T}$ where

$\mathcal{E} = \{e_1, e_2\}$

$\mathcal{B} = \{1, t, t^2\}$

b) $T(1) = \begin{bmatrix} 1+1 \\ 1 \end{bmatrix} = 2e_1 + e_2$

$T(t) = \begin{bmatrix} 0+1 \\ -1 \end{bmatrix} = 1 \cdot e_1 + -1 e_2$

$T(t^2) = \begin{bmatrix} 0+1 \\ 1 \end{bmatrix} = 1 \cdot e_1 + 1 \cdot e_2$

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{T} = \begin{bmatrix} | & | & | \\ [T(1)]_{\mathcal{E}} & [T(t)]_{\mathcal{E}} & [T(t^2)]_{\mathcal{E}} \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Example

Let $V = \text{Span}\{\sin t, \cos t\}$,

and $D: V \to V$ be the

linear transformation

$$D(f) = f'.$$

With $\mathcal{B} = \{\sin t, \cos t\}$,

find $\underset{\mathcal{B} \leftarrow \mathcal{B}}{D}$.

Example

Let $\mathcal{B} = \{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\}$.

Let $V = \operatorname{span} \mathcal{B}$.

Let $D: V \to V$ be the linear transformation

$$D(f) = f'.$$

Find $\underset{\mathcal{B} \leftarrow \mathcal{B}}{D}$, and use this to compute

$$\int 4e^{2x} - 3e^{2x}\sin x \, dx$$

$$D(e^{2x}) = 2e^{2x}$$

$$D(e^{2x}\cos x) = 2e^{2x}\cos x + -e^{2x}\sin x$$

$$D(e^{2x}\sin x) = 2e^{2x}\sin x + e^{2x}\cos x.$$

$$\underset{\mathcal{B} \leftarrow \mathcal{B}}{D} = \left[\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & -1 & 2 \end{array}\right].$$

If $I : V \to V$ is

$$I(f) = \int_0^x f(t)\,dt$$

then $\underset{B\leftarrow B}{I} = \left(\underset{B\leftarrow B}{D}\right)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{5}\cdot 2 & \frac{1}{5}\cdot(-1) \\ 0 & \frac{1}{5}\cdot 1 & \frac{1}{5}\cdot 2 \end{bmatrix}$

$$\left[\int 4e^{2x} - 3e^{2x}\sin x\,dx\right]_B = \underset{B\leftarrow B}{I} \cdot \left[4e^{2x} - 3e^{2x}\sin x\right]_B$$

$$= \underset{B\leftarrow B}{I} \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ \frac{3}{5} \\ -\frac{6}{5} \end{bmatrix}.$$

$$\int 4e^{2x} - 3e^{2x}\sin x\,dx$$

$$= 2e^{2x} + \frac{3}{5}e^{2x}\cos x + -\frac{6}{5}e^{2x}\sin x + C.$$