Contents

17 Complex Numbers Addendum– Lay Appendix B

17.1 Standard Form

Basic facts about the set $\mathbb{C}$ of complex numbers are:

1. The set $\mathbb{C}$ contains an element, usually called i, which is not a real number.
2. Every member of $\mathbb{C}$ can be written uniquely in the form
$$x + iy.$$
where x and y are real numbers.
(Here 'uniqueness' means that, if $x_1 + iy_1$ and $x_2 + iy_2$ represent the same complex number, then $x_1 = x_2$ and $y_1 = y_2$.)
3. The operations of addition, negation and multiplication are defined:
$$\begin{aligned} (x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2), \\ -(x_1 + iy_1) &= (-x_1) + i(-y_1), \\ (x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 - y_1y_2) \\ &\qquad + i(x_1y_2 + x_2y_1). \end{aligned}$$
From these basic facts all else follows.

17.2 Variations

There are various fairly obvious variations on standard form, For example:

- $x + yi$ (the i written after the y instead of before it).
- yi or iy for $0 + iy$.
- x for $x + i0$.
- $x - iy$ for $x + i(-y)$.
- i and $-i$ for $0 + 1i$ and $0 + (-1)i$.

and so on.

17.3 Square root of -1

The definition of multiplication gives:

$$i^2 = (0 + 1i)^2 = -1.$$

so i is the notorious 'square root' of -1'.

Note that also $(-i)^2 = -1$.

17.4 Arithmetic Operations

Theorem 17.1 *The set of complex numbers forms a field. This means that there is a zero element* $0 = 0{+}0i$ *and a multiplicative identity* $1 = 1{+}0i$. *The only trick is to find the inverse of any nonzero complex number.*

Here it is: If $x + iy \neq 0$ then

$$(x+iy)^{-1} = \left(\frac{x}{x^2+y^2}\right) - i\left(\frac{y}{x^2+y^2}\right).$$

Check that it is indeed the inverse of $x + iy$.

17.5 Real and Imaginary Parts

Definition 17.1
If $z = x + iy$, *then* x *is the real part, Re(z), and* y *is the imaginary part, Im(z).*

Definition 17.2
A complex number of the form iy *(*y *real) is called imaginary or purely imaginary.*

17.6 Inverses

To remember the formula for inverses, remember that

$$(x + iy)(x - iy) = x^2 + y^2.$$

We proceed in a manner similar to rationalising the denominator: multiply top and bottom by $(x - iy)$:

$$\begin{aligned}\frac{1}{x+iy} &= \frac{1}{x+iy}\,\frac{x-iy}{x-iy}\\ &= \frac{x-iy}{x^2+y^2}\\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\end{aligned}$$

17.7 Finding Roots

Finding the roots may be a difficult or impossible problem for some polynomials. For some special kinds of polynomials, the roots can be found without too much trouble. Quadratics can be factorised and their roots found by the good old

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

formula. This includes the quadratics whose roots include complex numbers.

17.8 Quadratic Formula Example

Example

Find the complex roots of the quadratic $x^2 + x + 1 = 0$.

The roots are

$$\frac{-1 \pm \sqrt{1^2 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

Call these roots α and β. We have

$$\alpha = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

and

$$\beta = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

and $x^2 + x + 1 = (x - \alpha)(x - \beta)$.

17.9 Conjugates

If $z = x + iy$ is a complex number, then its conjugate is the number $\overline{z}$ given by

$$\overline{z} = x - iy.$$

Example

The conjugate of $4 + 2i$ is $4 - 2i$.

The conjugate of $3 - 6i$ is $3 + 6i$.

17.10 Some Notation

$$\text{cis}\ \theta = \cos\theta + i\sin\theta = e^{i\theta}$$

This number represents a number that lies on the unit circle, at an angle of θ to the real axis.

Note $e^{i\pi} = -1$.

The number $r = \sqrt{x^2 + y^2}$ is called the absolute value of $x = x + iy$ and is written $|z|$.