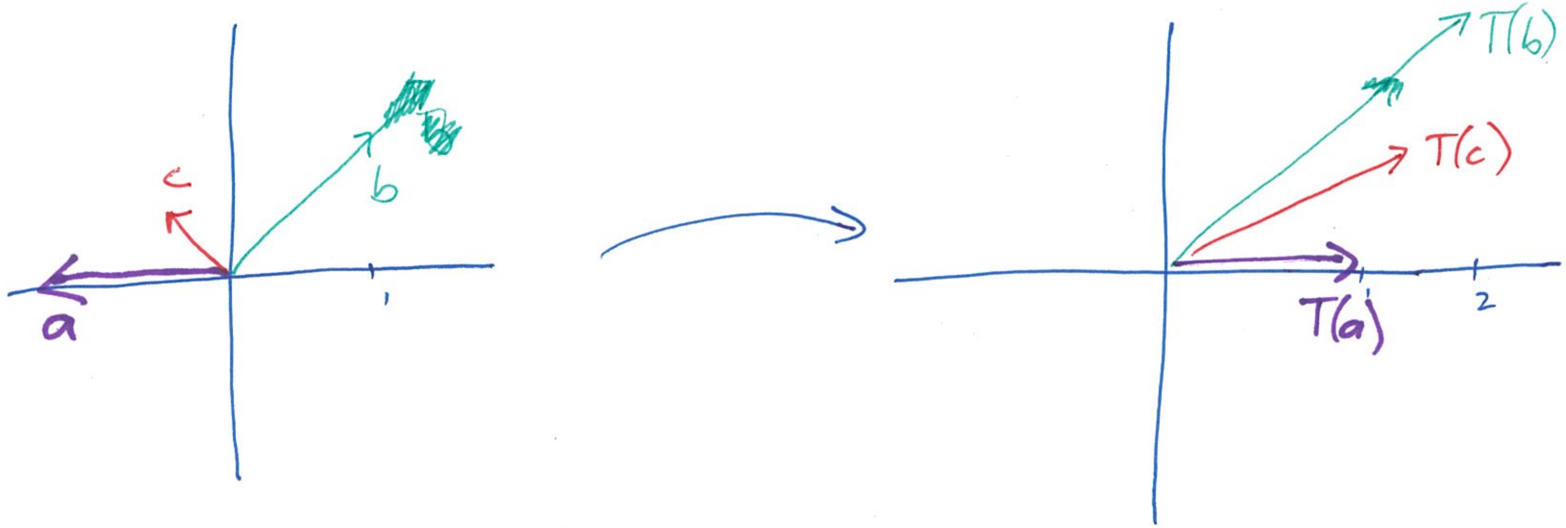
Suppose a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ acts as follows: ①



Write a matrix for T with respect to a basis of your choice.

(Bonus — use this to write the matrix for T with respect to the standard basis.)

Notice $T(b) = 2b$ and $T(a) = -a$.

Thus a and b are eigenvectors so make a convenient basis for describing T: Let $\mathcal{B} = \{a, b\}$, then

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{T} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

We could find $\underset{\mathcal{E}\leftarrow\mathcal{E}}{T}$ as $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\ \underset{\mathcal{B}\leftarrow\mathcal{B}}{T}\ \underset{\mathcal{B}\leftarrow\mathcal{E}}{P}$.

Perhaps (picking a scale) $\cancel{a = \begin{bmatrix} -1 \\ 0 \end{bmatrix}_{\mathcal{E}},\ b=}$ $[a]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $[b]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

so $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P} = \left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right)^{-1} = -1 \cdot \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

so $\underset{\mathcal{E}\leftarrow\mathcal{E}}{T} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}$.

What happens when the characteristic polynomial has <u>complex roots</u>?

~~Even~~ When we start with a real matrix, the coefficients of the characteristic polynomial must be real, but the roots may be complex.

We'll mostly concentrate on the 2×2 case.

Example

Let $A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$.

What are the roots of the characteristic polynomial?

Can you describe A geometrically?

We calculate $\det(A-\lambda I) = \begin{vmatrix} \cos\varphi - \lambda & -\sin\varphi \\ \sin\varphi & \cos\varphi - \lambda \end{vmatrix}$

$$= (\cos\varphi - \lambda)^2 + (\sin\varphi)^2$$

$$= \cos^2\varphi - 2\cos\varphi\,\lambda + \lambda^2 + \sin^2\varphi$$

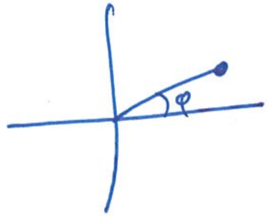
$$= \lambda^2 - 2\cos\varphi\,\lambda + 1$$

The roots are thus

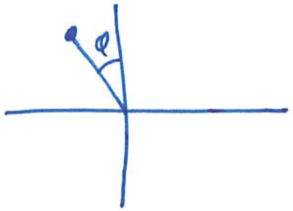
$$\frac{2\cos\varphi \pm \sqrt{4\cos^2\varphi - 4}}{2} = \cos\varphi \pm \sqrt{\cos^2\varphi - 1}$$

$$= \boxed{\cos\varphi \pm i\sin\varphi}$$

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix} =$$



$$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\varphi\\ \cos\varphi\end{bmatrix} =$$



so A is rotating counterclockwise through φ.

We see A cannot have any real eigenvectors unless

$$\varphi = 0, \pi, 2\pi, 3\pi, \ldots$$

But there are still complex eigenvectors!

$$E_{\cos\varphi + i\sin\varphi} = \mathrm{Nul}\begin{bmatrix} -i\sin\varphi & -\sin\varphi \\ \sin\varphi & -i\sin\varphi \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$ (6)

$$= \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -i \end{bmatrix}\right\}$$

Similarly

$$E_{\cos\varphi - i\sin\varphi} = \mathrm{Nul}\begin{bmatrix} i\sin\varphi & -\sin\varphi \\ \sin\varphi & -i\sin\varphi \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ i \end{bmatrix}\right\}.$$

Example

Find the eigenvectors of $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

The characteristic polynomial is

$$\det \begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda)+2 = \lambda^2 - 8\lambda + 17,$$

so the eigenvalues are

$$\lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm i.$$

(Complex eigenvalues of real matrices always come in conjugate pairs.)

$$E_{4+i} = \mathrm{Nul}(A - (4+i)I) = \mathrm{Nul}\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}.$$

Row reduction with complex coefficients is a bit unpleasant.

However we know this matrix must have a non-trivial nullspace. Since it's 2×2, we can take a shortcut: the nullspace is the space of solutions of

$$(1-i)x_1 - 2x_2 = 0$$

(as the 2nd equation will be redundant).

Thus $E_{4+i} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| (1-i)x_1 - 2x_2 = 0 \right\}$

$$= \left\{ \begin{bmatrix} x_1 \\ \frac{1-i}{2} x_1 \end{bmatrix} \middle| x_1 \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 \\ \frac{1-i}{2} \end{bmatrix} \right\}.$$

(Similarly for the other eigenspace —
and in fact the other eigenvector will be the complex conjugate of this one!)

Consider $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

The characteristic polynomial is $(a-\lambda)^2 + b^2 = a^2 - 2a\lambda + \lambda^2 + b^2$;

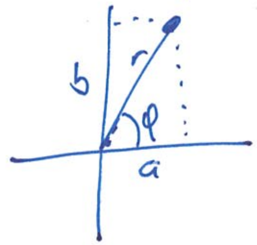
and so the eigenvalues of C are $a \pm ib$.

(This generalises our first example.)

In fact (assuming a and b are not both zero), define $r = \sqrt{a^2+b^2}$, and write

$$C = r\begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

where $\varphi = \cos^{-1}\frac{a}{r}$



Matrices of this form are a composition of a rotation $\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ and a dilation $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$.

Example

Describe the geometric action of $C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$.

$$C = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \sqrt{2} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}.$$

So C rotates everything by $\frac{\pi}{4}$, and then scales everything up by a factor of $\sqrt{2}$.

Theorem

Let A be a real 2×2 matrix with complex eigenvalue $\lambda = a - ib$, ~~then~~ and eigenvector $v \in \mathbb{C}^2$

$$A = PCP^{-1}, \text{ where } P = \begin{bmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{bmatrix}, \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

(This isn't quite diagonalisation, but it's similar. Notice here P and C are real matrices again.

Example

Describe the action of $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$ on $\mathbb{R}^2$ geometrically

$\det(A - \lambda I) = (2-\lambda)(-\lambda) + 2 = \lambda^2 - 2\lambda + 2,$

so the complex eigenvalues are

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

$$E_{1\mp i} = \mathrm{Nul}(A - (1 \mp i)I) =$$

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, (1+i)x_1 + x_2 = 0 \right\}$$

$$= \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \mp i \end{bmatrix} \right\}.$$

Applying the theorem, we have

$$a = b = 1, \quad v = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}, \quad \mathrm{Re}\, v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathrm{Im}\, v = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

so $A = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}^{-1}$

We can think of this as

$$A = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} \quad \underset{\mathcal{B} \leftarrow \mathcal{B}}{T} \quad \underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$$

where $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$.

Thus in the basis $\mathcal{B}$, T is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$= \sqrt{2} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$$

i.e. rotation by $\frac{\pi}{4}$ followed by scaling by $\sqrt{2}$.

Of course, in the standard basis A is more complicated!