Yesterday:

If A is a 2×2 matrix with real entries,
and a complex eigenvalue $a-ib$
and a corresponding (complex) eigenvector v,

then

$$A = PCP^{-1}$$

where $P = \begin{pmatrix} | & | \\ \mathrm{Re}\,v & \mathrm{Im}\,v \\ | & | \end{pmatrix}$, $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$,

for $r = \sqrt{a^2+b^2}$
$\varphi = \cos^{-1}\frac{a}{r}$.

Today: discrete linear dynamical systems.

The state of the system at time k is described by a vector $x_k \in \mathbb{R}^n$.
The evolution of the system is described by

$$x_{k+1} = A x_k,$$

for some matrix A.

What is the long term behaviour?

Let's assume A is diagonalisable
(perhaps with complex eigenvalues, but we'll do the real case first).

Thus A has linearly independent eigenvectors

$$v_1, \dots, v_n$$

and corresponding eigenvalues

$$\lambda_1, \dots, \lambda_n.$$

Let's reorder these if necessary so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

Any initial condition for our system x_0 can be written in the basis of eigenvectors:

$$x_0 = c_1 v_1 + \cdots + c_n v_n$$

This eigenvector decomposition determines the whole forward evolution allowing us to describe the sequence $\{x_k\}$.

If $$x_0 = c_1 v_1 + \cdots + c_n v_n$$

$$x_1 = A x_0 = c_1 A v_1 + \cdots + c_n A v_n$$
$$= c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n$$

$$x_2 = A x_1 = c_1 \lambda_1 A v_1 + \cdots + c_n \lambda_n A v_n$$
$$= c_1 (\lambda_1)^2 v_1 + \cdots + c_n (\lambda_n)^2 v_n$$

and in general

$$x_k = c_1 (\lambda_1)^k v_1 + \cdots + c_n (\lambda_1)^k v_n.$$

Example

Consider a forest full of owls and rats.
The owls and rats each mate and reproduce,
but the owls also eat the rats!

Let's consider a <u>linear</u> model for how the populations change over time:

measured in thousands

$$O_{k+1} = (0.5)\, O_k + (0.4)\, R_k$$

$$R_{k+1} = -p\, O_k + (1.1)\, R_k$$

for some positive parameter p, which we'll study.

(How realistic is this?
In real life the 'interaction terms' probably wouldn't be linear.
Nevertheless you can often linearize around a particular state, and obtain very good approximations!)

Example 1 $p = 0.104$, so $A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$.

The eigenvalues are $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$

with eigenvectors $v_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Any initial population x_0 can be written as

$$x_0 = c_1 v_1 + c_2 v_2$$

Then $x_k = c_1(1.02)^k v_1 + c_2(0.58)^k v_2$

$$= c_1(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2(0.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

As $k \to \infty$, $(0.58)^k \to 0$. Assume $c_1 > 0$.

Then for large k, $x_k \approx c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix}$

and $x_{k+1} \approx c_1 (1.02)^{k+1} \begin{bmatrix} 10 \\ 13 \end{bmatrix} \approx (1.02)\, x_k$.

Thus there about 10 owls for every 13 000 rats, and both populations grow by 2% each time step.

In general, if we have $|\lambda_1| \geqslant 1$,

and $1 > |\lambda_j|$ for all $j \geqslant 2$,

we see this behaviour: $x_{k+1} \approx \lambda_1 x_k \approx c_1 \lambda_1^{k+1} v_1$

Example 2 $\quad p = 0.2$ (a higher predation rate),

$$A = \begin{bmatrix} 0.5 & 0.4 \\ -0.2 & 1.1 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 0.9, \quad \lambda_2 = 0.7$

~~and~~ with eigenvectors: $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$

Thus $\quad x_k = c_1 (0.9)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} \xrightarrow{k \to \infty} 0.$

The owls eat too many rats, and everything dies out!

Example 3 $p = 0.125$, $A = \begin{bmatrix} 0.5 & 0.4 \\ -0.125 & 1.1 \end{bmatrix}$

Now the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0.6$

with eigenvectors $V_1 = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Thus $x_k = c_1 \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} + c_2 (0.6)^k \begin{bmatrix} 4 \\ 1 \end{bmatrix} \longrightarrow c_1 \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}$

Thus the population reaches an equilibrium,

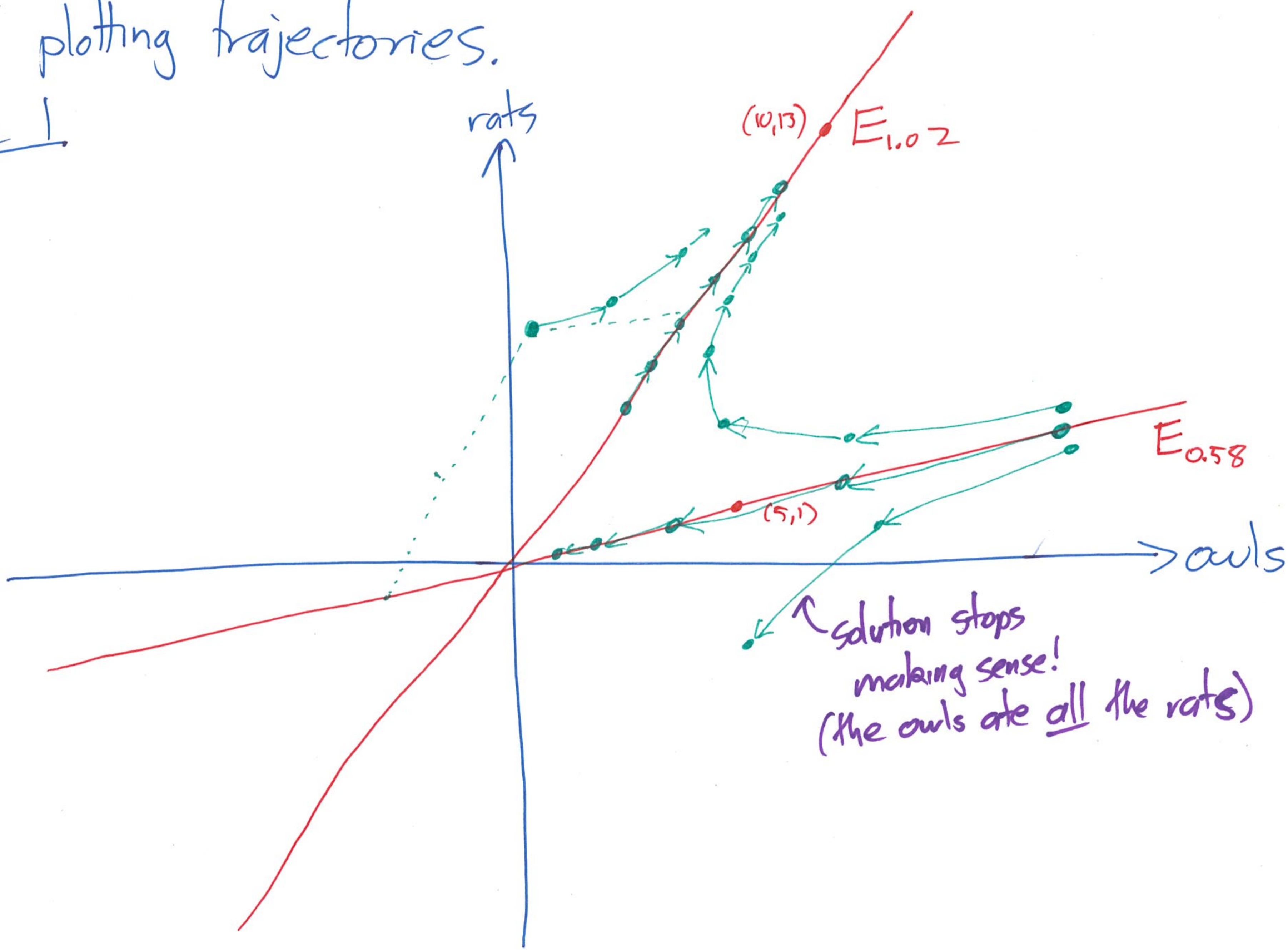
with 8 owl per 10000 rats,

and the overall population size depends only on the initial conditions.

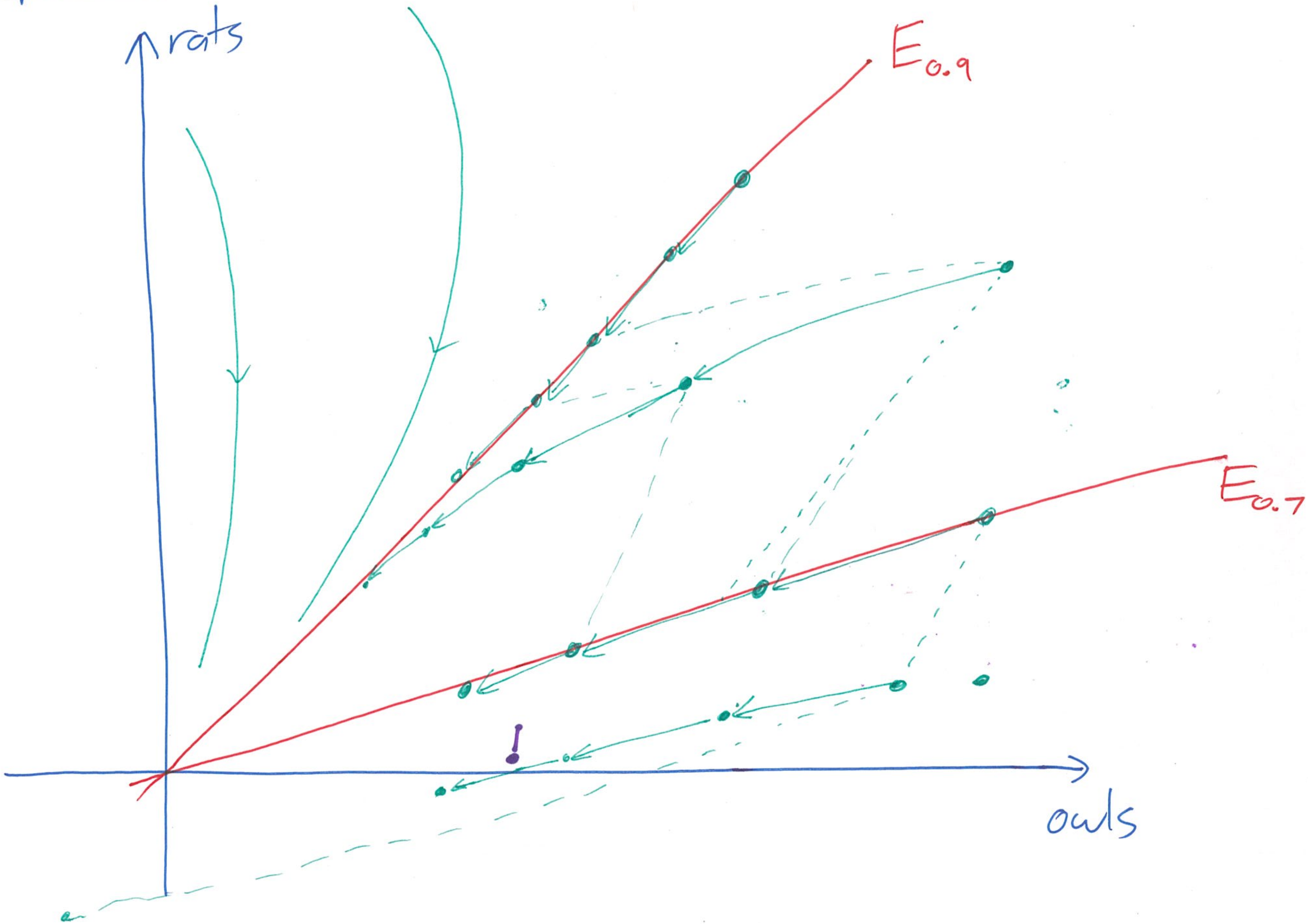
This equilibrium is unstable as we've seen — slight changes in the predation rate result in collapse or exponential growth.

We can describe all these solutions graphically, by plotting trajectories.

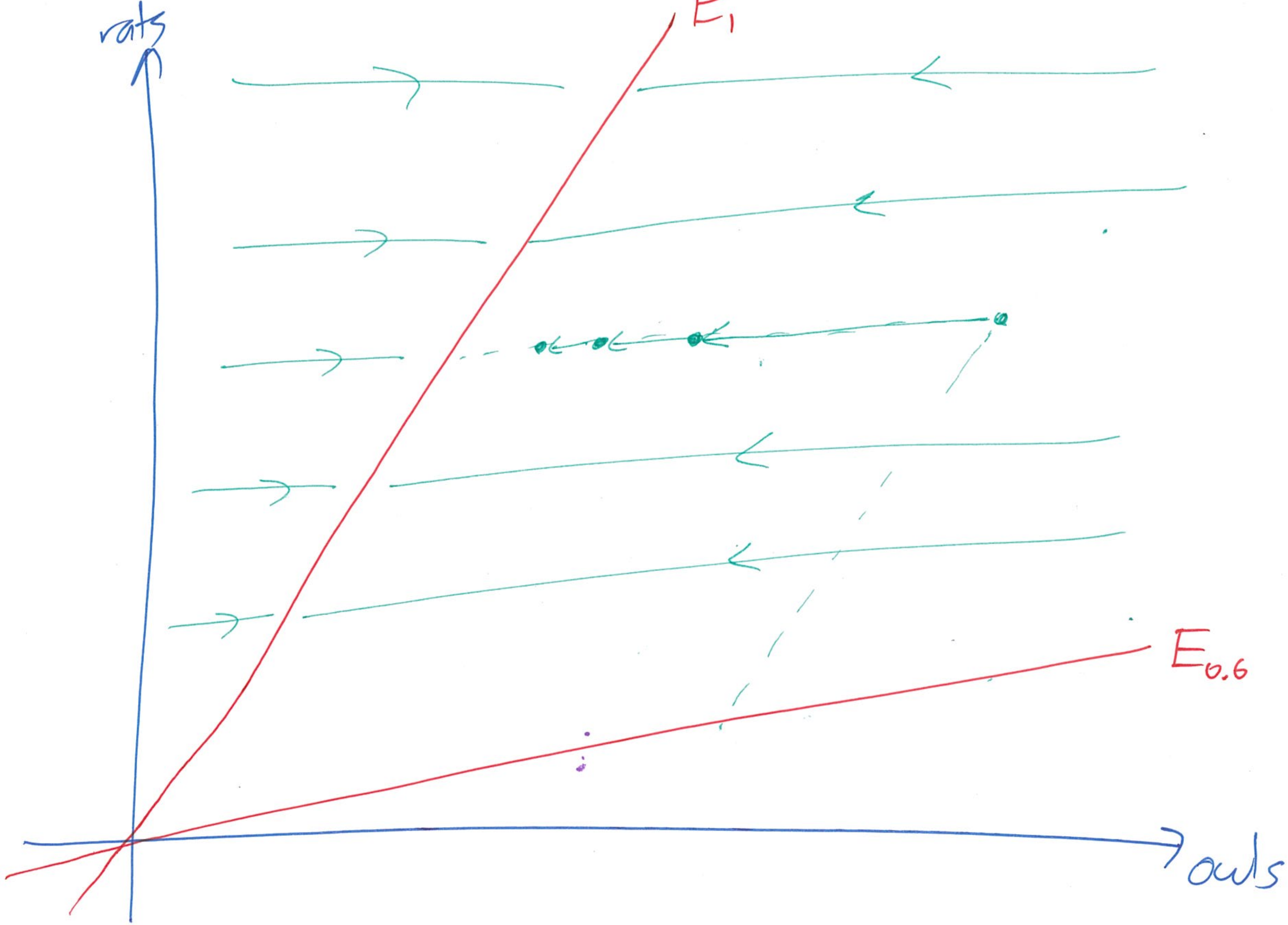
Example 1



Example 2



Example 3



Example 4

$p = 0.325$

$$\lambda_1 = 0.8 + 0.2i$$

$$v_1 \approx \begin{bmatrix} 0.743 \\ 0.557 + 0.371i \end{bmatrix}$$

so $A = PCP^{-1}$ where $P = \begin{bmatrix} 0.743 & 0 \\ 0.557 & 0.371 \end{bmatrix}$

$$C = \begin{bmatrix} 0.8 & 0.2 \\ -0.2 & 0.8 \end{bmatrix} \approx \begin{bmatrix} 0.825 & 0 \\ 0 & 0.825 \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

$$\varphi \approx 0.245 = 14^\circ$$

In[8]:= {{0.5, 0.4}, {-p, 1.1}} // Eigenvalues

Out[8]= $\left\{0.2\left(4.-\sqrt{2.25-10.\,p}\right),\ 0.2\left(4.+\sqrt{2.25-10.\,p}\right)\right\}$

In[9]:= 2.25` - 10.` p == -1 // Solve

Out[9]= {{p → 0.325}}

In[17]:= (A = {{0.5, 0.4}, {-0.325, 1.1}}) // Eigenvalues

Out[17]= {0.8 + 0.2 i, 0.8 - 0.2 i}

In[11]:= {{0.5, 0.4}, {-0.325, 1.1}} // Eigenvectors

Out[11]= {{0.742781 + 0. i, 0.557086 + 0.371391 i}, {0.742781 + 0. i, 0.557086 - 0.371391 i}}

In[12]:= {0.7427813527082073` + 0.` i, 0.5570860145311556` - 0.3713906763541035` i} // Re

Out[12]= {0.742781, 0.557086}

In[13]:= {0.7427813527082073` + 0.` i, 0.5570860145311556` - 0.3713906763541035` i} // Im

Out[13]= {0., -0.371391}

In[14]:= $0.8^2 + 0.2^2$ // Sqrt

Out[14]= 0.824621

In[16]:= $\dfrac{\text{ArcCos}[0.8 / 0.8246211251235323`]}{2\pi}\,360$

Out[16]= 14.0362

In[27]:= ListPlot[Table[MatrixPower[A, k].{0, 1}, {k, 0, 50}], PlotRange → All]

Out[27]=