Next up: inner products, and <u>approximate</u> solutions to linear systems.

Often $Ax=b$ has no solution.

Perhaps this is because the equations aren't quite right (experimental error measuring some parameters?) and we decide to look for an <u>approximate</u> solution x, such that <u>Ax is as close as possible to b</u>.

Since distance is the square root of a sum of squares, this is called a <u>least squares solution</u>.

<u>Recall</u> The dot (or scalar or inner) product of two vectors

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{in } \mathbb{R}^n$$

is the scalar

$$(u,v) = u \cdot v = u^T v = [u_1 \cdots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1 v_1 + \cdots + u_n v_n.$$

The following properties are easy to check:

1) $u \cdot v = v \cdot u$

2) $u \cdot (v+w) = u \cdot v + u \cdot w$

3) $k(u \cdot v) = (ku) \cdot v = u \cdot (kv)$ for $k \in \mathbb{R}$

4) $u \cdot u \geq 0$

and $u \cdot u = 0$ if and only if $u = 0$

Example

Let

$$u = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 4 \end{bmatrix}$$

$$v = \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$$

What is $u \cdot v$?

$$u \cdot v = (1)(-1) + (3)(0) + (-2)(3) + (4)(-2)$$

$$= -1 + 0 + -6 + -8$$

$$= -15$$

In $\mathbb{R}^3$, we can use the dot product to calculate the length of a vector:

$$\|u\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

In general, we just <u>define</u> the length of a vector by this formula!

In $\mathbb{R}^n$, $\|u\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + \cdots + u_n^2}$.

It follows that $\|cu\| = \sqrt{(cu)\cdot(cu)} = \sqrt{c^2(u \cdot u)} = |c|\,\|u\|$, so the length of cu is $|c|$ times the length of u.

A vector whose length is 1 is called a unit vector.

If v is a non-zero vector, then

$$u = \frac{v}{\|v\|}$$

is a unit vector in the direction of v.

Replacing v by the unit vector $\frac{v}{\|v\|}$ is called normalising v.

Let's check: $\|u\|^2 = u \cdot u = \frac{v}{\|v\|} \cdot \frac{v}{\|v\|}$

$$= \frac{1}{\|v\|^2}(v \cdot v) = \frac{\|v\|^2}{\|v\|^2} = 1.$$

Example

What is the normalisation of $u = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$?

First we calculate the length:

$$\|u\| = \sqrt{u \cdot u} = \sqrt{\begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}} = \sqrt{1+9+4} = \sqrt{14}.$$

Therefore

$$\frac{u}{\|u\|} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ 0 \\ \frac{2}{\sqrt{14}} \end{bmatrix}.$$

Vectors u and v are called orthogonal if $u \cdot v = 0$.

- Since $0 \cdot v = 0$, for every vector $v \in \mathbb{R}^n$, the zero vector is orthogonal to every vector.

- If u and v are orthogonal, so are $c_1 u$ and $c_2 v$, for any scalars c_1 and c_2.

- Because $u \cdot u \geq 0$, with equality <u>only</u> if $u = 0$, the only vector orthogonal to itself is the zero vector.

Definition

Suppose W is a subspace of $\mathbb{R}^n$.
If a vector z is orthogonal to every $w \in W$,
then we say z is orthogonal to W.

Example

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal

to $\text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Example

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is orthogonal to

$\text{Nul} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

Do this as an exercise!

Definition

The set of all vectors x that are orthogonal to W is called the orthogonal complement of W.

We write it as $W^\perp$. ("W perp")

$$W^\perp = \{x \in \mathbb{R}^n \mid x \cdot y = 0 \quad \forall y \in W\}.$$

- $x \in W^\perp$ iff x is orthogonal to every vector in a spanning set for W.
- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- $W \cap W^\perp = 0$, since 0 is the only vector orthogonal to itself.

Example

Find a basis for $V^\perp$, where

$$V = \operatorname{span}\left\{\begin{bmatrix}1\\3\\3\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\-1\\3\end{bmatrix}\right\}$$

$V^\perp$ consists of all vectors $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ in $\mathbb{R}^4$ satisfying

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix} \cdot \begin{bmatrix}1\\3\\3\\1\end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix}a\\b\\c\\d\end{bmatrix} \cdot \begin{bmatrix}3\\-1\\-1\\3\end{bmatrix} = 0$$

This gives a homogeneous system of two equations in four variables:

$$a + 3b + 3c + d = 0$$
$$3a - b - c + 3d = 0.$$

Row-reducing the augmented matrix we get

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 1 & 0 \\ 3 & -1 & -1 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

so c and d are free variables, and the general solution is

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -d \\ -c \\ c \\ d \end{bmatrix} = d\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

These two vectors are linearly independent, so a basis for $V^\perp$ is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Notice we found the orthogonal complement as the <u>nullspace</u> of a matrix:

$$V^{\perp} = \mathrm{Nul}\, A$$

$$A = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 3 & -1 & -1 & 3 \end{bmatrix}$$

Here the <u>rows</u> of A are the spanning set for V.

<u>Theorem</u> Let A be an $m \times n$ matrix.

The orthogonal complement of the row space of A is the same as the null space of A: $(\mathrm{Row}A)^{\perp} = \mathrm{Nul}A$.

(And hence, by taking transposes,

$$(\mathrm{Col}A)^{\perp} = \mathrm{Nul}(A^T), \text{ also.})$$

<u>Proof</u> If $x \in \mathrm{Nul}A$, x is orthogonal to each row, so

$$x \in (\mathrm{Row}A)^{\perp}.$$

Converseley, if $x \in (\mathrm{Row}A)^{\perp}$, x is orthogonal to each row, so

$$x \in \mathrm{Nul}A.$$

Thus $\mathrm{Nul}A = (\mathrm{Row}A)^{\perp}$.

Example

Calculate
Row A, Nul A, Col A, Nul A^T
for $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$

$$\text{Row}\,A = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

$$\text{Nul}\,A = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$$\text{Col}\,A = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

$$\text{Nul}\,A^T = \text{Nul}\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -1 & -2 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}.$$

Theorem If W is a subspace of $\mathbb{R}^n$, then

$$\dim W + \dim W^\perp = n.$$

Proof Define $A = \begin{bmatrix} \text{---} & w_1^T & \text{---} \\ \text{---} & w_2^T & \text{---} \\ & \vdots & \\ \text{---} & w_k^T & \text{---} \end{bmatrix}$

where $\{w_i\}_{i=1}^k$ is a basis for W.

Then $W = \text{Row} A$ and $W^\perp = (\text{Row} A)^\perp = \text{Nul} A$.

Now $\dim W = \dim \text{Row} A = \text{rank} A$,

and $\dim W^\perp = \dim \text{Nul} A$.

By the rank theorem

$$\dim W + \dim W^\perp = \text{rank} A + \dim \text{Nul} A = n.$$

Basis theorem

If $\mathcal{B} = \{b_1, \dots, b_m\}$ is a basis for $W \subset \mathbb{R}^{m+r}$

and $\mathcal{C} = \{c_1, \dots, c_r\}$ is a basis for $W^\perp$,

then $\mathcal{B} \cup \mathcal{C} = \{b_1, \dots, b_m, c_1, \dots, c_r\}$ is a basis for $\mathbb{R}^{m+r}$.

As a consequence, if W is a subspace of $\mathbb{R}^n$,
for any vector $v \in \mathbb{R}^n$ there is a <u>unique</u> way to write

$$v = w + u,$$

where $w \in W$ and $u \in W^\perp$.

Example

Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Decompose $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ as a sum of vectors in W and $W^\perp$.

We're given a basis for W, and we find

$$W^\perp = \text{Nul}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

Then $$\begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + (-1)\cdot\begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix},$$

so $$\begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix},$$ where $\begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} \in W$, $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \in W^\perp$.