Last time:

- $u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$
- u is orthogonal to v if $u \cdot v = 0$
- u is orthogonal to the subspace W if $u \cdot w = 0$ for all $w \in W$.
- the orthogonal complement of a subspace W is the subspace $W^\perp = \{x \in V \mid x \cdot w = 0 \text{ for all } w \in W\}$

<u>Theorem</u> $(\text{Row } A)^\perp = \text{Nul } A$.

<u>Corollary</u> If $V \subset \mathbb{R}^n$, $\dim V + \dim V^\perp = n$.

Today: a set of vectors is <u>orthogonal</u> if the elements are pairwise orthogonal.

Example.

Consider

$$u_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}.$$

Show that $\{u_1, u_2, u_3\}$ is an orthogonal set.

We need to verify that

$$u_1 \cdot u_2 = 0, \quad u_1 \cdot u_3 = 0, \quad \text{and} \quad u_2 \cdot u_3 = 0.$$

$$u_1 \cdot u_2 = -3 + -6 + -3 + 12 = 0$$

etc.

Theorem

If $S = \{v_1, \dots, v_k\}$ is an orthogonal set of nonzero vectors, then S is a linearly independent set.

(and hence a basis for span S)

Proof Suppose $c_1 v_1 + \cdots + c_k v_k = 0$ for some scalars c_i.

Then
$$
\begin{aligned}
0 &= (c_1 v_1 + \cdots + c_k v_k) \cdot v_1 \\
&= c_1(v_1 \cdot v_1) + \cancel{c_2(v_2 \cdot v_1)}^{\,0} + \cdots + \cancel{c_k(v_k \cdot v_1)}^{\,0} \\
&= c_1(v_1 \cdot v_1).
\end{aligned}
$$

Since v_1 is nonzero, $v_1 \cdot v_1$ is nonzero, so we conclude $c_1 = 0$.

A similar argument shows that each of the c_i are zero.

$\square$

Definition

An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthogonal set.

Example

Given

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix},$$

find a nonzero vector

$$u_4 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ so}$$

$\{u_1, u_2, u_3, u_4\}$ forms an orthogonal set.

(and hence a basis for $\mathbb{R}^4$!)

We are looking for a vector satisfying

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0,$$

i.e. a sol^n to the linear homogeneous equations

$$\begin{aligned} a + 2b + c &= 0 \\ a - b + c + 3d &= 0 \\ 2a - b - d &= 0, \end{aligned}$$

i.e something in the null space of

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & -1 \end{bmatrix}.$$

(Notice we are looking for something in $\{u_1, u_2, u_3\}^\perp = (\text{Row} A)^\perp = \text{Nul} A$.)

row-reducing, $\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$,

so the general solution has d free, and $a=b=d$, $c=-3d$.

These solutions are spanned by $\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$, and we can

use that as our u_4.

Theorem

Let $\{v_1, \dots, v_k\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$.

Then for any vector $w \in W$, the unique scalars $c_1, \dots, c_k$ so that $w = c_1 v_1 + \cdots + c_k v_k$ are given by the formula

$$c_i = \frac{w \cdot v_i}{v_i \cdot v_i}.$$

(That sure beats row-reducing!)

<u>Proof</u> Since we have a basis,

$$w = c_1 v_1 + \cdots + c_k v_k \quad \text{for some scalars } c_i.$$

To compute c_1, we take the inner product with v_1:

$$w \cdot v_1 = c_1 (v_1 \cdot v_1) + c_2 (\cancelto{0}{v_2 \cdot v_1}) + \cdots + c_k (\cancelto{0}{v_k \cdot v_1})$$

$$= c_1 (v_1 \cdot v_1)$$

and rearranging:

$$c_1 = \frac{w \cdot v_1}{v_1 \cdot v_1}.$$

This works for all the c_i, giving our formula.

Example

Given the orthogonal basis

$$\mathcal{U} = \left\{ \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$, find the $\mathcal{U}$-coordinates of

$$x = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}.$$

Exercise: verify $\mathcal{U}$ is actually orthogonal.

We compute

$$x \cdot u_1 = 12 - 6 + 0 = 6$$

$$x \cdot u_2 = 8 + 4 + 1 = 13$$

$$x \cdot u_3 = 4 + 2 - 4 = 2$$

$$u_1 \cdot u_1 = 18, \quad u_2 \cdot u_2 = 9, \quad u_3 \cdot u_3 = 18.$$

Then
$$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{x \cdot u_3}{u_3 \cdot u_3} u_3$$

$$= \frac{6}{18} u_1 + \frac{13}{9} u_2 + \frac{2}{18} u_3$$

$$= \frac{1}{3} u_1 + \frac{13}{9} u_2 + \frac{1}{9} u_3,$$

So
$$[x]_{\mathcal{U}} = \begin{bmatrix} \frac{1}{3} \\ \frac{13}{9} \\ \frac{1}{9} \end{bmatrix}.$$

<u>Definition</u> A set $\{u_1, \ldots, u_k\}$ in $\mathbb{R}^n$ is an

orthonormal set

if it is an orthogonal set of unit vectors.

The simplest example is the standard basis of $\mathbb{R}^n$.

You can take any set of nonzero orthogonal vectors, and normalize them all, to obtain an orthonormal set.

(They're still pairwise orthogonal after normalizing.)

Suppose $\{u_1, u_2, u_3\}$ is an orthonormal set in $\mathbb{R}^3$, and let $U = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix}$. Let's calculate

$$U^T U = \begin{bmatrix} - & u_1^T & - \\ - & u_2^T & - \\ - & u_3^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & u_1 \cdot u_3 \\ u_2 \cdot u_1 & u_2 \cdot u_2 & u_2 \cdot u_3 \\ u_3 \cdot u_1 & u_3 \cdot u_2 & u_3 \cdot u_3 \end{bmatrix}$$

(by orthogonality)

$$= \begin{bmatrix} u_1 \cdot u_1 & 0 & 0 \\ 0 & u_2 \cdot u_2 & 0 \\ 0 & 0 & u_3 \cdot u_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(unit vectors!)

Thus $U^T U = I$, and so $UU^T = I$ as well.

Definition a square matrix U with orthonormal colums is called an orthogonal matrix.

Theorem orthogonal matrices are automatically invertible, and $U^{-1} = U^T$.

Theorem
- an $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.
- in that case, for any vectors x, y
  - $\|Ux\| = \|x\|$
  - $(Ux) \cdot (Uy) = x \cdot y$

Example

Consider the 4×3 matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}.$$

It has orthogonal columns, and $A^TA = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

- Are the rows of A orthogonal?
- Obtain B by multiplying each column of A by a scalar, so $B^TB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- the rows are <u>not</u> orthogonal:

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = 2-1-2 = -1 \neq 0.$$

- we should normalize the columns!

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = 6, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} = 12, \quad \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} \cdot 6,$$

so $$B = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & 0 \\ 0 & \frac{3}{\sqrt{12}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

will do.