Overview

Last time we introduced the notion of an orthonormal basis for a subspace. We also saw that if a square matrix U has orthonormal columns, then U is invertible and $U^{-1} = U^T$. Such a matrix is called an *orthogonal* matrix.

At the beginning of the course we developed a formula for computing the projection of one vector onto another in $\mathbb{R}^2$ or $\mathbb{R}^3$. Today we'll generalise this notion to higher dimensions.

From Lay, §6.3

Review

Recall from Stewart that if $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then

$\text{proj}_{\mathbf{u}}\mathbf{y} = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$.

(Lay uses the notation " $\hat{\mathbf{y}}$ " for this projection, where $\mathbf{u}$ is understood.)

How would you describe the vector $\text{proj}_{\mathbf{u}}\mathbf{y}$ in words?
One possible answer:

> $\mathbf{y}$ *can be written as the sum of a vector parallel to* $\mathbf{u}$ *and a vector orthogonal to* $\mathbf{u}$*;* $\text{proj}_{\mathbf{u}}\mathbf{y}$ *is the summand parallel to* $\mathbf{u}$.

Or alternatively,

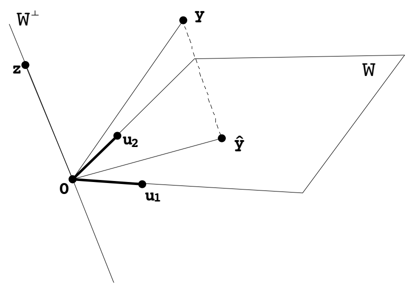
> $\mathbf{y}$ *can be written as the sum of a vector in the line spanned by* $\mathbf{u}$ *and a vector orthogonal to* $\mathbf{u}$*;* $\text{proj}_{\mathbf{u}}\mathbf{y}$ *is the summand in Span*$\{\mathbf{u}\}$.

We'd like to generalise this, replacing Span$\{\mathbf{u}\}$ by an arbitrary subspace:

Given $\mathbf{y}$ and a subspace W in $\mathbb{R}^n$, we'd like to write $\mathbf{y}$ as a sum of a vector in W and a vector in $W^\perp$.

Example 1

Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ and let $W = \text{Span}\ \{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector $\hat{\mathbf{y}}$ in W and a vector $\mathbf{z}$ in $W^\perp$.



Recall that for any orthogonal basis, we have

$$\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3.$$

It follows that

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2$$

and

$$\mathbf{z} = \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3.$$

Since $\mathbf{u}_3$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$, its scalar multiples are orthogonal to Span$\{\mathbf{u}_1, \mathbf{u}_2\}$. Therefore $\mathbf{z} \in W^{\perp}$

All this can be generalised to any vector $\mathbf{y}$ and subspace W of $\mathbb{R}^n$, as we will see next.

The Orthogonal Decomposition Theorem

Theorem

Let W be a subspace in $\mathbb{R}^n$. Then each $\mathbf{y} \in \mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \tag{1}$$

where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$.

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is any orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \tag{2}$$

*The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of $\mathbf{y}$ onto** W.*

Note that it follows from this theorem that to calculate the decomposition $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, it is enough to know one orthogonal basis for W explicitly. Any orthogonal basis will do, and all orthogonal bases will give the same decomposition $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$.

Example 2

Given

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

let W be the subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Write $\mathbf{y} = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}$ as the sum of a vector in W and a vector orthogonal to W.

The orthogonal projection of $\mathbf{y}$ onto W is given by

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 \\
&= \frac{-2}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + \frac{7}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + \frac{6}{3}\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix}5\\-8\\13\\3\end{bmatrix}
\end{aligned}$$

Also

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix}2\\-3\\4\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}5\\-8\\13\\3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\-1\\-1\\0\end{bmatrix}$$

Thus the desired decomposition of $\mathbf{y}$ is

$$\begin{aligned}
\mathbf{y} &= \hat{\mathbf{y}} + \mathbf{z} \\
\begin{bmatrix}2\\-3\\4\\1\end{bmatrix} &= \frac{1}{3}\begin{bmatrix}5\\-8\\13\\3\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\-1\\-1\\0\end{bmatrix}.
\end{aligned}$$

The Orthogonal Decomposition Theorem ensures that $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$ is in $W^\perp$. However, verifying this is a good check against computational mistakes.

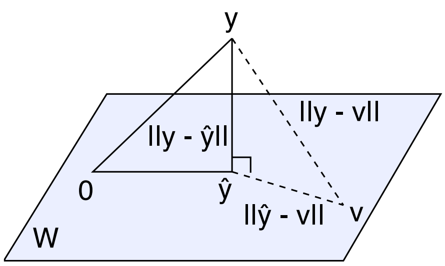
This problem was made easier by the fact that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for W. If you were given an arbitrary basis for W instead of an orthogonal basis, what would you do?

Theorem (The Best Approximation Theorem)

Let W be a subspace of $\mathbb{R}^n$, $\mathbf{y}$ any vector in $\mathbb{R}^n$, and $\hat{\mathbf{y}}$ the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest vector in W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \tag{3}$$

for all $\mathbf{v}$ in W, $\mathbf{v} \neq \hat{\mathbf{y}}$.



Proof
Let $\mathbf{v}$ be any vector in W, $\mathbf{v} \neq \hat{\mathbf{y}}$. Then $\hat{\mathbf{y}} - \mathbf{v} \in W$. By the Orthogonal Decomposition Theorem, $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to W. In particular $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to $\hat{\mathbf{y}} - \mathbf{v}$. Since

$$\mathbf{y} - \mathbf{v} = (\mathbf{y} - \hat{\mathbf{y}}) + (\hat{\mathbf{y}} - \mathbf{v})$$

the Pythagorean Theorem gives

$$\|\mathbf{y} - \mathbf{v}\|^2 = \|\mathbf{y} - \hat{\mathbf{y}}\|^2 + \|\hat{\mathbf{y}} - \mathbf{v}\|^2.$$

Hence $\|\mathbf{y} - \mathbf{v}\|^2 > \|\mathbf{y} - \hat{\mathbf{y}}\|^2$. □

We can now define the distance from a vector $\mathbf{y}$ to a subspace W of $\mathbb{R}^n$.

Definition

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{y}$ be a vector in $\mathbb{R}^n$. The *distance* from $\mathbf{y}$ to W is

$$||\mathbf{y} - \hat{\mathbf{y}}||$$

where $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto W.

Example 3

Consider the vectors

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

Find the closest vector to $\mathbf{y}$ in $W = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\begin{aligned} \hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\ &= \frac{30}{10}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \frac{26}{26}\begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}. \end{aligned}$$

Therefore the distance from $\mathbf{y}$ to W is $\left\| \begin{bmatrix} 3 \\ -1 \\ \vdots \end{bmatrix} - \begin{bmatrix} -1 \\ -5 \\ \vdots \end{bmatrix} \right\| = \left\| \begin{bmatrix} 4 \\ 4 \\ \vdots \end{bmatrix} \right\| = 8.$

Theorem

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then for all $\mathbf{y}$ in $\mathbb{R}^n$ we have

$$proj_W \mathbf{y} = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p.$$

This theorem is an easy consequence of the usual projection formula:

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p.$$

When each $\mathbf{u}_i$ is a unit vector, the denominators are all equal to 1.

Theorem

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for W and $U = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_p\end{bmatrix}$, then for all $\mathbf{y}$ in $\mathbb{R}^n$ we have

$$proj_W \mathbf{y} = UU^T\mathbf{y}\,. \tag{4}$$

The proof is a matrix calculation; see the posted slides for details.

Note that if U is a $n \times p$ matrix with orthonormal columns, then we have $U^TU = I_p$ (see Lay, Theorem 6 in Chapter 6). Thus we have

$U^TU\mathbf{x} = I_p\mathbf{x} = \mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^p$

$UU^T\mathbf{y} = \text{proj}_W\mathbf{y}$ for every $\mathbf{y}$ in $\mathbb{R}^n$, where $W = \text{Col } U$.

Note: Pay attention to the sizes of the matrices involved here. Since U is $n \times p$ we have that U^T is $p \times n$. Thus U^TU is a $p \times p$ matrix, while UU^T is an $n \times n$ matrix.

The previous theorem shows that the function which sends $\mathbf{x}$ to its orthogonal projection onto W is a linear transformation. The kernel of this transformation is ...

...the set of all vectors orthogonal to W, i.e., $W^\perp$.

The range is W itself.

The theorem also gives us a convenient way to find the closest vector to $\mathbf{x}$ in W: find an orthonormal basis for W and let U be the matrix whose columns are these basis vectors. Then mutitply $\mathbf{x}$ by UU^T.

Examples

Example 4

Let $W = \text{Span}\left\{\begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\2\\1\end{bmatrix}\right\}$ and let $\mathbf{x} = \begin{bmatrix}4\\8\\1\end{bmatrix}$. What is the closest vector to $\mathbf{x}$ in W?

Set $\mathbf{u}_1 = \begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$,

$$U = \begin{bmatrix}2/3 & -2/3\\1/3 & 2/3\\2/3 & 1/3\end{bmatrix}.$$

We check that $U^TU = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, so U has orthonormal columns.

The closest vector is

$$\text{proj}_W\mathbf{x} = UU^T\mathbf{x} = \frac{1}{9}\begin{bmatrix}8 & -2 & 2\\-2 & 5 & 4\\2 & 4 & 5\end{bmatrix}\begin{bmatrix}4\\8\\1\end{bmatrix} = \begin{bmatrix}2\\4\\5\end{bmatrix}.$$

We can also compute distance from $\mathbf{x}$ to W:

$$\|\mathbf{x} - \text{proj}_W\mathbf{x}\| = \|\begin{bmatrix}4\\8\\1\end{bmatrix} - \begin{bmatrix}2\\4\\5\end{bmatrix}\| = \|\begin{bmatrix}2\\4\\-4\end{bmatrix}\| = 6.$$

Because this example is about vectors in $\mathbb{R}^3$, so we could also use cross products:

$$\begin{bmatrix}2\\1\\2\end{bmatrix} \times \begin{bmatrix}-2\\2\\1\end{bmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & 1 & 2\\-2 & 2 & 1\end{vmatrix} = -3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k} = \mathbf{n}$$

gives a vector orthogonal to W, so the distance is the length of the projection of $\mathbf{x}$ onto $\mathbf{n}$:

$$\begin{bmatrix}4\\8\\1\end{bmatrix} \cdot \begin{bmatrix}-1/3\\-2/3\\2/3\end{bmatrix} = -6,$$

and the closest vector is

$$\begin{bmatrix}4\\8\\1\end{bmatrix} + 6\begin{bmatrix}-1/3\\-2/3\\2/3\end{bmatrix} = \begin{bmatrix}2\\4\\5\end{bmatrix}.$$

This example showed that the standard matrix for projection to $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$ is $\frac{1}{9}\begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$.

If we instead work with $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}$ coordinates, what is the orthogonal projection matrix?

Observe that the three basis vectors were chosen very carefully: $\mathbf{b}_1$ and $\mathbf{b}_2$ span W, and $\mathbf{b}_3$ is orthogonal to W. Thus each of the basis vectors is an eigenvector for the linear transformation. (Why?)

The linear transformation is represented by a diagonal matrix when it's written in terms of an eigenbasis. Thus we get the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

What does this tell you about orthogonal projection matrices in general?

Example 5

$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ are orthogonal and span a subspace W of $\mathbb{R}^4$. Find a vector orthogonal to W.

Normalize the columns and set

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/2 \\ 0 & 1/2 \\ 1/\sqrt{2} & -1/2 \\ 0 & -1/2 \end{bmatrix}.$$

Then the standard matrix for the orthogonal projection is has matrix

$$UU^T = \frac{1}{4}\begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 3 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}.$$

Thus, choosing a vector $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ not in W, the closest vector to $\mathbf{v}$ in W is given by

$$UU^T \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 \\ 2 \\ 1 \\ -2 \end{bmatrix}.$$

In particular, $\mathbf{v} - UU^T\mathbf{v} = \begin{bmatrix}3\\2\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}5\\2\\1\\-2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\2\\-1\\4\end{bmatrix}$ lies in $W^{\perp}$.

Thus $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\\4\end{bmatrix}$ are orthogonal in $\mathbb{R}^4$, and span a subspace W_1 of dimension 3.

But now we can repeat the process with W_1! This time take

$$U = \begin{bmatrix}1/\sqrt{2} & 1/2 & 1/\sqrt{22}\\ 0 & 1/2 & 2/\sqrt{22}\\ 1/\sqrt{2} & -1/2 & -1/\sqrt{22}\\ 0 & -1/2 & 4/\sqrt{22}\end{bmatrix},$$

$$UU^T = \frac{1}{44}\begin{bmatrix}35 & 15 & 9 & -3\\ 15 & 19 & -15 & 5\\ 9 & -15 & 35 & 3\\ -3 & 5 & 3 & 43\end{bmatrix}.$$

Taking $\mathbf{x} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, $(I_4 - UU^T)\mathbf{x} = 1/44\begin{bmatrix}3\\-5\\-3\\1\end{bmatrix}$ and then

$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\\4\end{bmatrix}, \begin{bmatrix}3\\-5\\-3\\1\end{bmatrix}$ is an orthogonal basis for $\mathbb{R}^4$. $\square$