Some Revision Questions, Solutions

1. Consider the following two bases for $\mathbb{R}^2$

$$\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \right\}.$$

Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Solution. The change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ is the matrix

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix}.$$

(To remember this formula keep in mind that you want to pass FROM $\mathcal{B}$-coordinates TO $\mathcal{C}$-coordinate so you need to know the $\mathcal{C}$ coordinates of the vectors $\mathbf{b}_1, \mathbf{b}_2$.)

Thus we have to find the coordinate vectors of $\mathbf{b}_1$ and $\mathbf{b}_2$ in the basis $\mathcal{C}$. For this we have to solve the two vector equations

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 = \mathbf{b}_1 \quad \text{(that will give the coordinate vector } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\mathbf{b}_1]_{\mathcal{C}})$$

and

$$y_1\mathbf{c}_1 + y_2\mathbf{c}_2 = \mathbf{b}_2 \quad \text{(that will give the coordinate vector } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [\mathbf{b}_2]_{\mathcal{C}}).$$

Each vector equation above gives a system of two linear equations in two variables. Since the two systems have the same coefficient matrix (the 2×2 matrix $[\ \mathbf{c}_1 \ \ \mathbf{c}_2\ ]$) we can solve these two systems at the same time by row reducing

$$\left[\begin{array}{cc|cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array} \right] = \left[\begin{array}{cc|cc} 2 & 6 & -6 & 2 \\ -1 & -2 & -1 & 0 \end{array} \right]$$

(the "new" basis is on the left and the "old" basis is on the right). We have

$$\left[\begin{array}{cc|cc} 2 & 6 & -6 & 2 \\ -1 & -2 & -1 & 0 \end{array} \right] \to \left[\begin{array}{cc|cc} 1 & 3 & -3 & 1 \\ -1 & -2 & -1 & 0 \end{array} \right] \to \left[\begin{array}{cc|cc} 1 & 3 & -3 & 1 \\ 0 & 1 & -4 & 1 \end{array} \right] \to \left[\begin{array}{cc|cc} 1 & 0 & 9 & -2 \\ 0 & 1 & -4 & 1 \end{array} \right]$$

Thus $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = [[\mathbf{b}_1]_{\mathcal{C}}\,[\mathbf{b}_2]_{\mathcal{C}}] = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$

2. Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & -2 & 8 \\ 0 & 5 & -2 \\ 0 & -4 & 3 \end{bmatrix}.$$

Determine if A is diagonalisable, and if so find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution

The characteristic equation for A is

$$\begin{aligned} 0 &= (3-\lambda)\{(5-\lambda)(3-\lambda) - 8\} \\ &= (3-\lambda)\{15 - 8\lambda + \lambda^2 - 8\} \\ &= (3-\lambda)\{7 - 8\lambda + \lambda^2\} \\ &= (3-\lambda)(7-\lambda)(1-\lambda) \end{aligned}$$

So the eigenvalues of A are 1, 3 and 7. Since A has three distinct eigenvalues, A is diagonalisable. To find the invertible matrix P we need to find the eigenspaces of A.

$$E_1 = \text{Nul }(A - I) = \text{Nul } \begin{bmatrix} 2 & -2 & 8 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \end{bmatrix}$$

$$= \text{Nul } \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} = \text{Nul } \begin{bmatrix} 1 & 0 & 7/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

So a basis for E_1 is $\begin{bmatrix} -7/2 \\ 1/2 \\ 1 \end{bmatrix}$ or to make calculation easier, $\begin{bmatrix} -7 \\ 1 \\ 2 \end{bmatrix}$.

$$E_3 = \text{Nul }(A - 3I) = \text{Nul } \begin{bmatrix} 0 & -2 & 8 \\ 0 & 2 & -2 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \text{Nul } \begin{bmatrix} 0 & 1 & -4 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \text{Nul } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for E_3 is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$E_7 = \text{Nul }(A - 7I) = \text{Nul }\begin{bmatrix} -4 & -2 & 8 \\ 0 & -2 & -2 \\ 0 & -4 & -4 \end{bmatrix}$$

$$= \text{Nul }\begin{bmatrix} 1 & 1/2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Nul }\begin{bmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for E_7 is $\begin{bmatrix} 5/2 \\ -1 \\ 1 \end{bmatrix}$, or to make calculation easier, $\begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$.

From these calculations the matrices P and D are

$$P = \begin{bmatrix} -7 & 1 & 5 \\ 1 & 0 & -2 \\ 2 & 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Note that the matrix D depends on the order in which the eigenvectors are used in P.

To check the answer we show that $AP = PD$:

$$AP = \begin{bmatrix} 3 & -2 & 8 \\ 0 & 5 & -2 \\ 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} -7 & 1 & 5 \\ 1 & 0 & -2 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 3 & 35 \\ 1 & 0 & -14 \\ 2 & 0 & 14 \end{bmatrix},$$

$$PD = \begin{bmatrix} -7 & 1 & 5 \\ 1 & 0 & -2 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} -7 & 3 & 35 \\ 1 & 0 & -14 \\ 2 & 0 & 14 \end{bmatrix}.$$

3. Find all the real values of k for which the matrix A is diagonalisable.

$$(i)\ A = \begin{bmatrix} 1 & k & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (ii)\ A = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

(i) Since A is an upper triangular matrix we can see that the eigenvalues of A are 1 and 2. For A to be diagonalisable we need

$$\dim E_1 + \dim E_2 = 3\,,$$

where E_1, E_2 are the eigenspaces associated with the eigenvalues 1 and 2 respectively. Since 2 has multiplicity 1 we know that $\dim E_2 = 1$. Since 1 has multiplicity 2 we know that $\dim E_1$ can be 1 or 2. Thus for A to be diagonalisable we need $\dim E_1 = 2$. We need to check the dimension of E_1.

$$E_1 = \text{Nul }(A - I) = \text{Nul }\begin{bmatrix} 0 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Nul }\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for any real number k. Since there are 2 free variables (x_1 and x_3), E_1 has dimension 2 for any real number, K and A is diagonalisable for any real number k.

(ii) Again A is upper triangular and has eigenvalue 1 with multiplicity 3. For A to be diagonalisable we need $\dim E_1 = 3$. So we check the dimension of the eigenspace E_1:

$$E_1 = \text{Nul }(A - I) = \text{Nul }\begin{bmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If k is non-zero the dimension of E_1 is 2, and A is not diagonalisable. If $k = 0$, E_1 has dimension 3 and A is diagonalisable.

4. Show that the matrices A and B are similar by showing that they are similar to the same diagonal matrix. Then find an invertible matrix P such that $P^{-1}AP = B$.

$$A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution

Since A is upper triangular we can see that the eigenvalues for A are 3 and -1.

The characteristic equation for B is

$0 = (1-\lambda)(1-\lambda) - 4 = 1 - 2\lambda + \lambda^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$

Thus the eigenvalues of B are 3 and -1 and both A and B are similar to the diagonal matrix $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

We now find an invertible matrix P_1 such that $A = P_1 D P_1^{-1}$ or equivalently $D = P_1^{-1} A P_1$. To do this we find the eigenvectors for A:

$$E_3 = \text{Nul } (A - 3I) = \text{Nul } \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} = \text{Nul } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

A basis for E_3 (for A) is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$E_{-1} = \text{Nul } (A + I) = \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}.$$

A basis for E_{-1} (for A) is $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$, and we can take $P_1 = \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix}$.

We now find an invertible matrix P_2 such that $B = P_2 D P_2^{-1}$ or equivalently $D = P_2^{-1} B P_2$. To do this we find the eigenvectors for B:

$$E_3 = \text{Nul } (B - 3I) = \text{Nul } \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = \text{Nul } \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

A basis for E_3 (for B) is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$E_{-1} = \text{Nul } (B + I) = \text{Nul } \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \text{Nul } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

A basis for E_{-1} (for B) is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and we can take $P_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

We now have that $D = P_1^{-1}AP_1$ and $D = P_2^{-1}BP_2$, so that

$$P_1^{-1}AP_1 = P_2^{-1}BP_2 \quad \text{or} \quad B = P_2P_1^{-1}AP_1P_2^{-1}.$$

So if we put $P = P_1P_2^{-1}$ then $P^{-1}AP = B$ as asked by the question. So

$$P = P_1P_2^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}$$

An easy check shows that $AP = PB$.

5. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

(a) Sketch the first six points of the trajectory for the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ taking $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. From this would you classify the the origin as a spiral attractor, spiral repellor, or orbital centre?

(b) Find an invertible matrix P and a matrix C of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$.

Solution

(a)

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

$$\mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$\mathbf{x}_5 = A\mathbf{x}_4 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

You can plot these points on an $x - y$ plane. We have $\mathbf{x}_0 = \mathbf{x}_6$, so any further points will just cycle around the points we have already. It should be clear that the trajectory doesn't spiral into the origin, or spiral away from the origin. Thus the origin is an orbital centre.

(b) To answer this part of the question we need to find the eigenvalues and eigenvectors of A. The characteristic equation is

$$0 = (1 - \lambda)(-\lambda) + 1 = \lambda^2 - \lambda + 1.$$

The roots of this equation are

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}.$$

(Note that the eigenvalues have modulus

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1\,.$$

This agrees with the fact that the origin is an orbital centre for the corresponding dynamical system, as we have observed in part (a).)

Take $\lambda = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ and find the corresponding eigenvector:

$$E_\lambda = \text{Nul }(A - \lambda I) = \text{Nul }\begin{bmatrix} 1/2 + i\sqrt{3}/2 & -1 \\ 1 & -1/2 + i\sqrt{3}/2 \end{bmatrix}$$

We can use either the first or second row of this matrix. The first row gives

$$(1/2 + i\sqrt{3}/2)x_1 - x_2 = 0 \quad \text{or} \quad x_2 = (1/2 + i\sqrt{3}/2)x_1.$$

This gives an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1/2 + i\sqrt{3}/2 \end{bmatrix}$. To make calculation easier we could take $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 + i\sqrt{3} \end{bmatrix}$.

Note that for $\lambda = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ the eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 - i\sqrt{3} \end{bmatrix}$.

The matrix P is

$$P = \begin{bmatrix} Re(\mathbf{v}_1) & Im(\mathbf{v}_1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & \sqrt{3} \end{bmatrix},$$

and the matrix C is

$$C = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}.$$

We can check the answer by showing that $AP = PC$:

$$AP = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{3} \\ 2 & 0 \end{bmatrix},$$

$$PC = \begin{bmatrix} 2 & 0 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{3} \\ 2 & 0 \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} -3 & 2 \\ -1 & -5 \end{bmatrix}$. Find the (complex) eigenvalues and a basis for each eigenspace.

Solution The characteristic polynomial is given by

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -3-\lambda & 2 \\ -1 & -5-\lambda \end{bmatrix} \\ &= (-3-\lambda)(-5-\lambda) + 2 \\ &= \lambda^2 + 8\lambda + 17. \end{aligned}$$

Solving $\lambda^2 + 8\lambda + 17 = 0$ gives

$$\lambda = \frac{-8 \pm \sqrt{64-68}}{2} = \frac{-8 \pm 2i}{2} = -4 \pm i.$$

Take the eigenvalue $\lambda = -4 - i$. To find an associated eigenvector we find the null space of $A - (-4-i)I$:

$$A + (4+i)I = \begin{bmatrix} 1+i & 2 \\ -1 & -1+i \end{bmatrix}$$

Recall that both rows of this matrix give the same information. I choose to use the second row so that I can avoid fractions in my answer. the information given is that

$$-x_1 + (-1+i)x_2 = 0 \quad \text{or} \quad x_1 = (-1+i)x_2.$$

We can choose an arbitrary value for x_2 and I choose $x_2 = 1$. This gives the eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

which is a basis for the corresponding eigenspace.

Now we know from the theory that $\overline{\mathbf{v}}_1$ (the conjugate of $\mathbf{v}_1$) is an eigenvector for the eigenvalue $\lambda = -4 + i$. So we can take

$$\mathbf{v}_2 = \overline{\mathbf{v}}_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}.$$

as a basis for the eigenspace corresponding to $-4 + i$.

Let us also find a factorisation $A = PCP^{-1}$, with C of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $a, b \in \mathbb{R}$.

Take the eigenvalue $\lambda = -4 - i$. The matrix P is given by

$$P = [\text{Re } \mathbf{v}_1 \ \ \text{Im } \mathbf{v}_1] = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

C comes directly from the eigenvalue:

$$C = \begin{bmatrix} -4 & -1 \\ 1 & -4 \end{bmatrix}.$$

We can check the correctness of P and C by showing that $AP = PC$:

$$AP = \begin{bmatrix} -3 & 2 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -4 & -1 \end{bmatrix},$$

$$PC = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -4 & -1 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -4 & -1 \end{bmatrix}.$$

7. Find the orthogonal projection of $\mathbf{v}$ onto the subspace W of $\mathbb{R}^4$ spanned by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

$$\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 4 \\ -3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find the distance from $\mathbf{v}$ to W.

Solution

Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, the projection $\hat{\mathbf{v}}$ is given by

$$\begin{aligned} \hat{\mathbf{v}} &= \frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{v}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{v}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 \\ &= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-2}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

The distance from $\mathbf{v}$ to W is

$$\|\mathbf{v} - \hat{\mathbf{v}}\| = \left\| \begin{bmatrix} 3 \\ -2 \\ 4 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 3 \\ -3 \\ 3 \\ -3 \end{bmatrix} \right\| = \sqrt{3^2 + 3^2 + 3^2 + 3^2} = 6.$$

8. Find all possible values of a, b in $\mathbb{R}$ for which the 2×2 matrix

$$U = \begin{bmatrix} a & \frac{2}{\sqrt{5}} \\ b & \frac{1}{\sqrt{5}} \end{bmatrix}$$

is orthogonal.

Solution

Recall that an orthogonal matrix is a square matrix with orthonormal columns. Thus we have to find all possible values of a, b in $\mathbb{R}$ for which the two columns

$$\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

form an orthonormal set.

Note that the second column has length one since $\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1.$

Requiring that the two columns are orthogonal to each other is the same as requiring that their dot product is zero, that is $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = 0$. This gives the linear equation

$$a\frac{2}{\sqrt{5}} + b\frac{1}{\sqrt{5}} = 0$$

that is equivalent to

$$(1) \qquad 2a + b = 0\,.$$

Imposing the condition that $\begin{bmatrix} a \\ b \end{bmatrix}$ has length one gives the degree two equation

$$(2) \qquad a^2 + b^2 = 1\,.$$

From (1) we get

$$b = -2a\,.$$

Substituting this into (2) we get $a^2 + 4a^2 = 1$ that is

$$a = \pm\frac{1}{\sqrt{5}}\,.$$

Thus there are only two possibilities for the pair a, b

$$a = \frac{1}{\sqrt{5}}, b = -\frac{2}{\sqrt{5}} \quad \text{and} \quad a = -\frac{1}{\sqrt{5}}, b = \frac{2}{\sqrt{5}}\,.$$

9. A dynamical system is described by the matrix equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ where the matrix A is given by

$$A = \begin{bmatrix} 0.5 & 0.2 \\ -0.5 & 1.2 \end{bmatrix}.$$

The matrix A has eigenvalues 1 and 0.7.

(a) Find the eigenvectors of A.

(b) If $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$, find the long term behaviour of the dynamical system.

Solution

(a) E_1 is the null space of $A - I$:

$$A - I = \begin{bmatrix} -0.5 & 0.2 \\ -0.5 & 0.2 \end{bmatrix} \to \begin{bmatrix} 1 & -0.4 \\ 0 & 0 \end{bmatrix}$$

so that $x_1 = 0.4x_2$. This gives $\mathbf{v}_1 = \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}$ as an eigenvector.

$E_{0.7}$ is the null space of $A - 0.7I$:

$$\begin{bmatrix} -0.2 & 0.2 \\ -0.5 & 0.5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

so that $x_1 = x_2$. This gives $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) The long term behaviour is given by

$$\mathbf{x}_k = \lambda_1^k c_1 \mathbf{v}_1 + \lambda_2^k c_2 \mathbf{v}_2$$

where $\lambda_1 = 1, \lambda_2 = 0.7$ and c_1 and c_2 are determined by $\mathbf{x}_0$ and satisfy:

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \quad \Longleftrightarrow \begin{bmatrix} 4 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 0.4 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To find c_1, c_2 we row reduce the augmented matrix of the above system:

$$\left[\begin{array}{cc|c} 0.4 & 1 & 4 \\ 1 & 1 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0.4 & 1 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0.6 & 1.2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right]$$

Thus $c_1 = 5, c_2 = 2$ and

$$\mathbf{x}_k = 1^k 5 \begin{bmatrix} 0.4 \\ 1 \end{bmatrix} + (0.7)^k 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to 5 \begin{bmatrix} 0.4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

as $k \to \infty$.

10. On any given day, a student is either healthy or ill. Of the students who are healthy today, 90% will be healthy tomorrow. Of the students who are ill today, 30% will be ill tomorrow.

   (a) Construct the stochastic matrix for this situation.

   (b) Suppose that 20% of the students are ill on Monday. What percentage of the students are likely to be ill on Wednesday?

   (c) In the long run what fraction of the students are expected to be healthy?

   Solution.

   (a) If we take the states as Healthy and Ill the transition matrix is given by

$$\begin{array}{cc} & \text{From:} \\ & \begin{array}{cc} \textit{Healthy} & \textit{Ill} \end{array} \quad \text{To:} \\ T = & \begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.3 \end{bmatrix} \begin{array}{c} \textit{Healthy} \\ \textit{Ill} \end{array} \end{array}$$

   (b) From the information in the question $\mathbf{x}_0 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ is the probability vector describing the situation on Monday. To find the percentage of students ill on Wednesday we need to find $\mathbf{x}_2$.

$$\begin{aligned} \mathbf{x}_2 = T\mathbf{x}_1 = T^2\mathbf{x}_0 &= \begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.3 \end{bmatrix}^2 \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \\ &= \begin{bmatrix} 0.88 & 0.84 \\ 0.12 & 0.16 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \\ &= \begin{bmatrix} 0.872 \\ 0.128 \end{bmatrix} . \end{aligned}$$

   So the proportion of students expected to be ill on Wednesday is 12.8%. *It is not actually necessary to calculate the proportion expected to be healthy (that is the first entry of $\mathbf{x}_2$), and in fact we could just calculate the second row of T^2 since we only need that row to calculate the second entry of $\mathbf{x}_2$, which is the proportion of students which is expected to be ill on Wednesday.*

   (c) Since the stochastic matrix T is regular[1] we know that for any ini-

[1] Recall that a stochastic matrix T is regular if some matrix power T^r has only strictly positive entries.

tial probability vector $\mathbf{x}_0$ the Markov chain $\mathbf{x}_k = T^k\mathbf{x}_0$ converges to the unique steady state vector of T (this means that the long term behaviour is described by the unique steady state vector of T, independently of the initial probability vector $\mathbf{x}_0$). Thus to answer this question we need to find the steady state vector, and to do this we find the null space of $T - I$:

$$T - I = \begin{bmatrix} -0.1 & 0.7 \\ 0.1 & -0.7 \end{bmatrix} \to \begin{bmatrix} 1 & -7 \\ 0 & 0 \end{bmatrix}$$

This gives $x_1 = 7x_2$, and putting this into the equation $x_1 + x_2 = 1$ we get $x_2 = 1/8$ and $x_1 = 7/8$ so that the steady state vector is $\mathbf{x} = \begin{bmatrix} 7/8 \\ 1/8 \end{bmatrix}$. So in the long run 7/8 of the students are expected to be well.

11. $T : M_{2\times 2} \to M_{2\times 2}$ is given by

$$T(A) = AB - BA$$

where $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Find the matrix of T with respect to the "standard" basis for $M_{2\times 2}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(b) Find a basis for the kernel of T.

(c) Explain why T is not one to one.

(d) Find a basis for the range of T.

(e) Explain why T is not onto.

Solution

(a) First note that

$$\begin{aligned} T(A) = AB - BA &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} - \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} -c & a-d \\ 0 & c \end{bmatrix} \end{aligned}$$

From this

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and has coordinate vector } \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and has coordinate vector } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$T\left(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right) = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \text{ and has coordinate vector } \begin{bmatrix}-1\\ 0\\ 0\\ 1\end{bmatrix},$$

$$T\left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix} \text{ and has coordinate vector } \begin{bmatrix}0\\ -1\\ 0\\ 0\end{bmatrix}.$$

Hence the matrix of T is

$$T_{\mathcal{B}} = \begin{bmatrix}0 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

(b) The kernel of T is the set of all matrices A for which $T(A) = 0$ (the zero matrix). So for matrices in the kernel of T:

$$\begin{bmatrix}-c & a-d\\ 0 & c\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

This gives $a = d$ and $c = 0$, while b is free. So matrices in $\ker(T)$ are of the form

$$\begin{bmatrix}a & b\\ 0 & a\end{bmatrix} = a\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + b\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},$$

and a basis for $\ker(T)$ is given by

$$\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right\}.$$

(c) T is not one to one as it has a non zero kernel.

(d) From our calculations the range of T consists of all matrices of the form $\begin{bmatrix}-c & a-d\\ 0 & c\end{bmatrix}$. We can write

$$\begin{bmatrix}-c & a-d\\ 0 & c\end{bmatrix} = a\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + c\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} + d\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}.$$

From this we can see that $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right\}$ span the range of T. However they are not linearly independent and so they don't form a basis. A basis for the range of T is given by $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(e) A matrix such as $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is not in the range of T. Hence T cannot be onto.

12. $T : \mathbb{P}_2 \to \mathbb{P}_2$ is given by

$$T(p(x)) = p(3x + 2).$$

(a) Find the matrix of T with respect to the standard basis for $\mathbb{P}_2$, $\mathcal{B} = \{1, x, x^2\}$.

(b) If possible find a basis for $\mathbb{P}_2$ for which the matrix of T is a diagonal matrix.

Solution

(a) To find the matrix of T we find the effect of T on the basis vectors $1, x, x^2$.

$$T(1) = 1, \quad T(x) = 3x + 2, \quad T(x^2) = (3x + 2)^2 = 4 + 12x + 9x^2.$$

we now find the coordinate vectors with respect to the standard basis.:

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [T(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, [T(x^2)]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 12 \\ 9 \end{bmatrix}.$$

The matrix of T with respect to B is given by

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 9 \end{bmatrix}$$

(b) To answer the second part of the question we aim first to diagonalise the matrix $[T]_{\mathcal{B}}$. When we have done that we can translate the information back to $\mathbb{P}_2$.

Since $[T]_{\mathcal{B}}$ is an upper triangular matrix we can read the eigenvalues on the diagonal: 1, 3 and 9. Because these are all distinct, we know that T is diagonalisable. We find the eigenspaces corresponding to these eigenvalues.

$$E_1 = \text{Nul }([T]_{\mathcal{B}} - I) = \text{Nul } \begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 12 \\ 0 & 0 & 8 \end{bmatrix} = \text{Nul } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives $x_2 = x_3 = 0$ and x_1 is free. So an eigenvector corresponding to $\lambda = 1$ is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$E_3 = \text{Nul}\ ([T]_{\mathcal{B}} - 3I) = \text{Nul}\ \begin{bmatrix} -2 & 2 & 4 \\ 0 & 0 & 12 \\ 0 & 0 & 6 \end{bmatrix} = \text{Nul}\ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives $x_1 = x_2$ and $x_3 = 0$, so an eigenvector for E_3 is given by $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$$E_9 = \text{Nul}\ ([T]_{\mathcal{B}} - I) = \text{Nul}\ \begin{bmatrix} -8 & 2 & 4 \\ 0 & -6 & 12 \\ 0 & 0 & 0 \end{bmatrix} = \text{Nul}\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives $x_1 = x_3$ and $x_2 = 2x_3$, o an eigenvector for E_3 is given by $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

So a basis in $\mathbb{R}^3$ for which $[T]_{\mathcal{B}}$ is diagonal is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$, and a basis $\mathcal{C}$ in $\mathbb{P}_2$ for which $[T]_{\mathcal{C}}$ is diagonal is $\mathcal{C} = \{1, 1+x, 1+2x+x^2\}$.

13. Consider the vector space W given by $W = \text{Span}\,\{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\}$. Let $D : W \to W$ be the differential operator defined by $D(f(x)) = f'(x)$ for every $f(x) \in W$ (where $f'(x)$ is the derivative of $f(x)$).

    (a) Find the matrix of D with respect to $\mathcal{B} = \{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\}$.

    (b) Compute the derivative of $f(x) = 3e^{2x} - 3e^{2x}\cos x + 5e^{2x}\sin x$ using the matrix you have just constructed in part (a).

    (c) Use the matrix in part (a) to find $\int (2e^{2x}\cos x - 4e^{2x}\sin x)\, dx$.

Solution

(a) We find the effect of D on the basis vectors:

$$D(e^{2x}) = 2e^{2x}, D(e^{2x}\cos x) = 2e^{2x}\cos x - e^{2x}\sin x,$$

$$D(e^{2x}\sin x) = 2e^{2x}\sin x + e^{2x}\cos x.$$

We now find the coordinate vectors:

$$[D(e^{2x})]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad [D(e^{2x}\cos x)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad [D(e^{2x}\sin x)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Hence the matrix $[D]_{\mathcal{B}}$ is given by

$$[D]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}.$$

(b) The coordinate vector for $f(x)$ is given by $[f(x)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}$. We calculate

$$[D]_{\mathcal{B}}[f(x)]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 13 \end{bmatrix}$$

This is the same as $[D(f(x))]_{\mathcal{B}}$ the coordinate vector of the derivative of $f(x)$ and so it tells us that the derivative of $f(x)$ is

$$f'(x) = 6e^{2x} - e^{2x}\cos x + 13e^{2x}\sin x\,.$$

(c) The coordinate vector of $g(x) = 2e^{2x}\cos x - 4e^{2x}\sin x$ in the basis $\mathcal{B}$ is

$$[g(x)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}.$$

The inverse of $[D]_{\mathcal{B}}$ is given by

$$[D]_{\mathcal{B}}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

We have

$$[D]_{\mathcal{B}}^{-1}[g(x)]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{8}{5} \\ -\frac{6}{5} \end{bmatrix}$$

that is

$$[D]_{\mathcal{B}} \begin{bmatrix} 0 \\ \frac{8}{5} \\ -\frac{6}{5} \end{bmatrix} = [g(x)]_{\mathcal{B}}.$$

The above identity tells us that

$$D\left(\frac{8}{5}e^{2x}\cos x - \frac{6}{5}e^{2x}\sin x\right) = g(x) = 2e^{2x}\cos x - 4e^{2x}\sin x,$$

so $\frac{8}{5}e^{2x}\cos x - \frac{6}{5}e^{2x}\sin x$ is an antiderivative of $2e^{2x}\cos x - 4e^{2x}\sin x$. Thus we have

$$\int 2e^{2x}\cos x - 4e^{2x}\sin x = \frac{8}{5}e^{2x}\cos x - \frac{6}{5}e^{2x}\sin x + C$$

where C is an arbitrary constant (check this result by differentiation!).