- Show that $\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$ is not diagonalisable

- Find an eigenvector for $\begin{pmatrix} 1 & 5 \\ -3 & 4 \end{pmatrix}$ and the corresponding (complex) eigenvalue.

- If we have factored a square matrix as $A = PDP^{-1}$, with D diagonal, explain what the eigenvalues and corresponding eigenvectors of A are.

- Write $\begin{pmatrix} 1 & 5 \\ -3 & 4 \end{pmatrix} = PCP^{-1}$, for real matrices P and C so P is invertible and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for some a and b.

- Find the QR decomposition of $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$.

- Find the least squares solution to $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $(R\hat{x} = Q^T b)$ $(A^T A \hat{x} = A^T b)$

- Consider $T: P_3 \to \mathbb{R}^3$, $\cancel{T(a+bx+cx^2+dx^3)}$

$$T(p) = \left(p(1),\ p'(0),\ p''(1)\right).$$

Find the matrix for T with respect to the bases

$$\mathcal{B} = \{1,\ 1+x,\ 1+x+x^2,\ 1+x+x^2+x^3\} \text{ for } P_3$$

and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$.

– What is the kernel of T? – Is T one-to-one? – Is T onto?

Suppose that in any given year I am either Healthy, Sick, or Dead.

If I was Healthy in a given year, 90% of the time I will be Healthy again next year, and 10% of the time I will be Sick next year.

Similarly, if I am Sick, 80% of the time I will be Sick again next year, and 20% of the time I will be Dead.

Once I'm Dead, I stay ~~Dea~~ Dead.

— Write down a stochastic matrix M and Markov chain describing this scenario.

— ~~This~~ M should have eigenvalues $1, \frac{4}{5}, \frac{9}{10}$, with corresponding eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$.

If I'm Healthy in ~~2010~~ 2015, what is the chance I'm not Dead in 2025?

Show $\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$ is not diagonalisable.

A matrix is diagonalisable if and only if ~~∃~~ there is a basis, consisting on eigenvectors.

We want to show that we can't make a basis out of the eigenvectors.

The eigenvalues are ... 2, because the matrix is triangular, so the eigenvalues are the diagonal entries

(or: compute the characteristic polynomial

$$\det\left(\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 2-\lambda & 0 \\ 2 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)^2$$

whose only is 2.)

The eigenvectors for 2 are just:

$$E_2 = \cancel{\text{Nul}(\ldots)}$$

$$\text{Nul}\left(\begin{bmatrix}2 & 0\\ 2 & 2\end{bmatrix} - 2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \text{Nul}\begin{bmatrix}0 & 0\\ 2 & 0\end{bmatrix}$$

$$= \text{Span}\left\{\begin{bmatrix}0\\ 1\end{bmatrix}\right\}.$$

all the linear combos of $\left\{\begin{bmatrix}0\\ 1\end{bmatrix}\right\}$

So all eigenvectors are multiples of $\begin{bmatrix}0\\ 1\end{bmatrix}$. i.e. $\text{span}\left\{\begin{bmatrix}0\\ 1\end{bmatrix}\right\}$

$$\parallel$$

$$\left\{x\begin{bmatrix}0\\ 1\end{bmatrix} : x \in \mathbb{R}\right\}$$

$$\parallel$$

$$\left\{\begin{bmatrix}0\\ x\end{bmatrix} : x \in \mathbb{R}\right\}$$

So we can't make a basis consisting of eigenvectors, e.g. because $\begin{bmatrix}1\\ 0\end{bmatrix}$ is not in the span of any eigenvectors.

Consider $T: P_3 \longrightarrow \mathbb{R}^3$

polynomials of degree at most 3

defined by $T(p) = \begin{bmatrix} p(1) \\ p'(0) \\ p''(1) \end{bmatrix}$

(e.g. $T(1+x^2) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$). Here $(1+x^2)'' = (2x)' = 2$

Find the matrix for T with respect to the bases

$$\mathcal{B} = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$$

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{T} = \begin{bmatrix} [T(b_1)]_{\mathcal{C}} & [T(b_2)]_{\mathcal{C}} & \cdots & [T(b_4)]_{\mathcal{C}} \end{bmatrix}$$

$$T(b_1) = T(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(b_2) = T(1+x) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$T(b_3) = T(1+x+x^2) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$(1+x+x^2)' = 1+2x$$

$$(1+x+x^2)'' = 2$$

$$T(b_4) = T(1+x+x^2+x^3) = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}.$$

$$(1+x+x^2+x^3)' = 1+2x+3x^2$$

$$(1+x+x^2+x^3)'' = 2+6x$$

$$T(b_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad T(b_1) = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$[T(b_1)]_e = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(b_2) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

so $[T(b_2)]_e = \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

$$T(b_3) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

here $a+b=1$

$a-b=2$

$2b+2=1$

$b=-\frac{1}{2}$

$a=\frac{3}{2}$

$$[T(b_3)]_e = \begin{bmatrix} 3 \\ \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

$$T(b_4) = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$a+b=1$
$a-b=8$

$a=b+8$
$2b+8=1$
$b=-\frac{7}{2}$
$a=\frac{9}{2}$

$$[T(b_4)]_e = \begin{bmatrix} 4 \\ \frac{9}{2} \\ -\frac{7}{2} \end{bmatrix}$$

so $$\underset{e \leftarrow B}{T} = \begin{bmatrix} | & | & & | \\ [T(b_1)]_e & [T(b_2)]_e & \cdots & [T(b_4)]_e \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{9}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{7}{2} \end{bmatrix}.$$

What is the kernel of T?

① compute the nullspace of $\underset{\mathcal{E}\leftarrow\mathcal{B}}{T}$

$$\text{Nul}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{9}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{7}{2} \end{bmatrix}$$

$$= \text{Nul}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & -2 & -8 \end{bmatrix}$$

so x_4 is free, $-2x_3 - 8x_4 = 0$, $x_3 = -4x_4$

$x_2 = -3x_3 - 9x_4$

$= 3x_4$

$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$

$x_1 = -6x_4 + 12x_4 - 4x_4 = 2x_4$

$$\operatorname{Nul} \underset{\mathcal{E}\leftarrow\mathcal{B}}{T} = \operatorname{Span}\left\{\begin{bmatrix}2\\3\\-4\\1\end{bmatrix}\right\}$$ these are coordinates with respect to the $\mathcal{B}$ basis.

That's the nullspace of the matrix associated to the linear transformation, in some crazy basis.

What is the kernel?

$$\operatorname{kernel}(T) = \operatorname{span}\{2(1) + 3(1+x) + -4(1+x+x^2) + 1(1+x+x^2+x^3)\}$$

$$= \operatorname{span}\{2 + -3x^2 + x^3\}.$$

$(2-3x^2+x^3)' = -6x+3x^2$

$(2-3x^2+x^3)'' = -6+6x$

Check: $T(2-3x^2+x^3) = \begin{bmatrix}2-3+1=0\\0\\0\end{bmatrix}$

Hooray!

(2) Find the kernel directly!

A typical element of P_3 is $a+bx+cx^2+dx^3$.

If $T(a+bx+cx^2+dx^3)=0$,

what can we say about a, b, c, d?

$$T(a+bx+cx^2+dx^3) = \begin{bmatrix} a+b+c+d \\ b \\ 2c+6d \end{bmatrix}.$$

$$(a+bx+cx^2+dx^3)'' = 2c+6dx$$

So $a+b+c+d=0$, $b=0$, $2c+6d=0$.

$\Rightarrow c=-3d \qquad a-3d+d=0$, so $a=2d$.

$$\ker T = \{2d + 0x + -3dx^2 + dx^3\}$$

$$= \operatorname{span}\{2 - 3x^2 + x^3\}$$

– Is T one-to-one?

A linear transformation is one-to-one
if and only if
it's kernel is just $\{0\}$.

So T is not one-to-one.

– Is T onto?

① We could directly show every element of $\mathbb{R}^3$
is in the image of T
<u>or</u> find an element of $\mathbb{R}^3$ not in the image.

E.g. if $\begin{bmatrix}1\\2\\3\end{bmatrix} \in \operatorname{im} T$, that means

$T(a+bx+cx^2+dx^3) = \begin{bmatrix}1\\2\\3\end{bmatrix}$ is consistent.

2 Remember:

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(P_3)$$

the source of the linear transform

$$\operatorname{rank}(T) = \dim(\text{image } T)$$

$$\operatorname{nullity}(T) = \dim(\ker T) = 1$$

$$\dim(P_3) = 4$$

$\therefore$ $\operatorname{rank}(T) = 3$

image $T \subset \mathbb{R}^3$

but $\dim(\text{image } T) = \dim \mathbb{R}^3$

So image T is all of $\mathbb{R}^3$

i.e. T is onto.

Find the QR decomposition of $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$

To find $A = QR$

we apply Gram-Schmidt to the columns of A.

$$v_1 = a_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

$$v_2 = a_2 - \frac{a_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix} - \frac{\frac{2}{3}}{\frac{5}{4}} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix} - \frac{8}{15} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{30} \\ \frac{2}{30} \end{bmatrix}.$$

$$V_3 = a_3 - \frac{a_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{a_3 \cdot V_2}{V_2 \cdot V_2} V_2$$

$$= \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} - \frac{\frac{1}{3} + \frac{1}{8}}{\frac{5}{4}} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} - \frac{-\frac{1}{90} + \frac{2}{120}}{\frac{5}{(30)^2}} \begin{bmatrix} -\frac{1}{30} \\ \frac{2}{30} \end{bmatrix}$$

So... let's pretend came out nicer:

- we still need to <u>normalize</u> those vectors, and assemble them as the columns of Q.

- to find R, use $R = Q^T A$

  and just multiply out.

Check: R should automatically be upper triangular.

To find a least squares solution $\hat{x}$ to $Ax = b$,

either solve

$$\boxed{A^T A \hat{x} = A^T b}$$

or $\boxed{R\hat{x} = Q^T b.}$

$A\hat{x}$ is as close as possible to b.

- The stochastic matrix is

$$M = \begin{pmatrix} 0.9 & 0.0 & 0.0 \\ 0.1 & 0.8 & 0.0 \\ 0.0 & 0.2 & 1.0 \end{pmatrix}$$

- $x_{2015} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We need to write this in the basis of eigenvectors.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \quad \text{so } c=-2,\ b=1,\ a=1$$

Then $x_{2025} = M^{10}x_{2015} = M^{10}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + M^{10}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + -2\,M^{10}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{4}{5}\right)^{10}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + (-2)\left(\frac{9}{10}\right)^{10}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Thus the chance I'm dead in 2025 is

$$1 + \left(\frac{4}{5}\right)^{10} - 2\left(\frac{9}{10}\right)^{10}.$$