Practice problems

Consider the matrix $A_t = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}$.

(1) Let S be the square with vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. What is the image of S under the linear transformation defined by multiplication by A_t? (Your answer will depend on t, of course.) What are the eigenvalues and eigenvectors of A_t?

(2) Recall that the matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates $\mathbb{R}^2$ by θ in the counterclockwise direction. (If you didn't remember this fact, how could you work it out?)

Describe the linear transformation whose standard matrix is $R_\theta A_t$. What does it do to S?

(3) Given a fixed vector $\mathbf{x}$ and a fixed value of t, there is a θ such that $\mathbf{x}$ is an eigenvector for $R_\theta A_t$. Writing out a formula for this is complicated, but see if you can work out a few examples (i.e., pick an $\mathbf{x}$ and a t and then find θ) to see why this is true.

(4) Now consider the linear transformation defined by $R_\theta A_t R_\theta^{-1}$. What are the eigenvalues and eigenvectors of this linear transformation? The kernel? The range?

(5) When $t = 0$, notice that A_t is the matrix for orthogonal projection to the x-axis. Use the results above to write down the standard matrix for orthogonal projection to the line $y = x$. (Your answer can be a product of matrices.) How can you check that you have the correct answer?

(6) If we take the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, what is the $\mathcal{B}$-coordinate matrix for orthogonal projection to the line $y = x$? What is the relationship between this matrix and the one you found in the previous part?

(7) What can you say about the dynamical system associated to each of the matrices you've considered so far?

(8) Can you find the standard matrix (possibly written as a product of other matrices) which has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for the eigenvalue 1 and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for the eigenvalue 2?

If I left "standard" off this question, what basis should you use to make the problem as simple as possible?

(9) Give a geometric description of the linear transformation represented by the standard matrix B, where

$$B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Linear Transformations

(1) Give an example of a function $f : \mathbb{R}^3 \to \mathbb{R}^3$ which is *not* a linear transformation.
(2) Give an example of a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is the xy-plane.
(3) Give an example of a linear transformation with the property that the dimension of its range is greater than the dimension of its kernel.
(4) Give an example of a linear transformation with the property that the dimension of its kernel is greater than the dimension of its range.
(5) Give an example of a linear transformation which has n orthogonal eigenvectors.

Matrices

(1) Give an example of a matrix which has orthogonal columns.
(2) Give an example of a matrix which has orthonormal columns.
(3) Give an example of a square matrix which is not diagonalisable.
(4) Give an example of a matrix A which satisfies $A^T = A^{-1}$.

Subspaces

(1) Show that the orthogonal complement of Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^4$.
(2) Find a subset of $\mathbf{R}^2$ which is closed under addition but not under scalar multiplication. (Of course, this means it's not a subspace!)
(3) Find a subset of $\mathbf{R}^2$ which is closed under scalar multiplication but not under addition. (Of course, this means it's not a subspace!)

A few tricky ones, just for fun

(1) Given a polynomial of the form $p(x) = x^2 + ax + b$, can you always find matrix which $p(x)$ as its characteristic polynomial? What about $q(x) = -x^3 + ax^2 + bx + c$? (This is a conceptual question, not a computational one.)
(2) Suppose that P is a linear transformation whose only eigenvalues are 0 and 1. If the 0-eigenspace is the orthogonal complement of the 1-eigenspace, show that P is orthogonal projection to the 1-eigenspace.

Eigenvectors The goal in this set of questions is to focus on the conceptual definition of an eigenvector, rather than the calculations often used to find them. Describing each linear transformation in geometric language makes it easier to determine which vectors are scaled, rather than rotated, by the transformation.

(1) S is stretched vertically by A_t. For example, if $t = 2$, then the image of S is the rectangle with vertices $(0,1)$, $(0,2)$, $(1,0)$, and $(1,2)$. If $t = 0$, S is collapsed to an interval on the x-axis. Since A_t is diagonal, the standard basis vectors are eigenvectors and the associated eigenvalues are the diagonal entries.

(2) Remember that $R_\theta A_t \mathbf{x}$ is the vector we get by first multiplying $\mathbf{x}$ by A_t and then by R_θ. Thus, S is stretched as in the previous part and then rotated in the counterclokwise direction.

(3) As an example, consider $\mathbf{x} = [1,1]^T$ and $t = 2$. Then $A_t\mathbf{x} = [1,2]^T$, so the new vector makes the angle $\frac{\pi}{3}$ with the x-axis. Applying R_θ for $\theta = \frac{-\pi}{12}$ rotates $A_t\mathbf{x}$ back to the line spanned by $\mathbf{x}$.

(4) Rotate $\mathbf{e_1}$ and $\mathbf{e_2}$ counterclockwise by θ to find the eigenvectors of $R_\theta A_t R_\theta^{-1}$. Even if you didn't come up with these vectors yourself, can you see why they're correct?

(5) $R_{\frac{\pi}{4}} A_0 R_{\frac{\pi}{4}}^{-1}$

(6) The $\mathcal{B}$ coordinate matrix is the diagonal matrix with 1 and 0 on the diagonal. Why? $[1,1]^T$ is an eigenvector with eigenvalue 1 and $[1,-1]^T$ is an eigenvector with eigenvalue 0. This matrix is similar to the one from the previous part.

(7) It's a boring dynamical system! Since the eigenvalues are 1 and 0, the system is static after the first time we apply the projection matrix.

(8) $R_{\frac{\pi}{4}} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} R_{\frac{\pi}{4}}^{-1}$. If you write this matrix in eigenvector coordinates instead of standard coordinates, it's just a diagonal matrix with 1 and 2 on the diagonal.

(9) The left-most matrix is R_θ for $\theta = \frac{\pi}{6}$. Therefore, B first rotates the plane clockwise by $\frac{\pi}{6}$, then stretches it vertically by a factor of three, and then rotates the stretched plane counterclockwise by $\frac{\pi}{6}$.

For any of the questions that ask for an example, there are many possible answers. If you're have trouble finding an example, it's okay to guess something and then check if it has the desired property –you don't need to know ahead of time if your guess is a good one as long as you can confirm/reject it.

Linear Transformations

(1) $f(x, y, z) = (1, 1, 1)$
(2) Orthogonal projection to the z-axis
(3) Multiplication by any matrix whose rank is greater than the dimension of its null space. For example, any matrix with trivial null space and nontrivial rank would work, such as [1] or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (Also, remember that the Rank Theorem tells us that the rank plus the dimension of the column space is equal to the number of columns.
(4) $[1 \ \ 1 \ \ 1]$
(5) The identity, or multiplication by any diagonal matrix (assuming standard coordinates).

Matrices

(1) The identity matrix, or $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
(2) The identity matrix, or $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
(3) $R_{\frac{\pi}{3}}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. For this one, it's great if you think, "Rotation by $\frac{\pi}{3}$ rotates any vector in the plane off itself, so there are no real eigenvectors, so it's not a diagonalisable linear transformation." However, if you don't think of this, you can write down some simple 2×2 matrices with 0 and 1 as the entries and just check to see which ones don't have two linearly independent eigenvectors.
(4) Any square matrix with orthonormal columns

Subspaces

(1) You can check this directly or show that this set is equal to the null space of [1 1 1 1].
(2) The positive quadrant of $\mathbb{R}^2$
(3) The x-axis and the y-axis in $\mathbb{R}^2$