Matrix Workshop Hints and Solutions

(1) **Warm-up** (Skip this one if you feel comfortable with the basic definitions and computations.)

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation whose standard matrix is $A = \begin{bmatrix} \frac{5}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{2} \end{bmatrix}$.

(a) Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{E}}$ an eigenvector for T? What about $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{E}}$?

(b) How would you find all the eigenvectors for T?

(c) What does the following equation tell you about T?

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{2} \end{bmatrix}$$

(For example, how does this equation identify the eigenvectors for T? The eigenvalues? What does T do to $\mathbb{R}^2$?)

(2) **A taste of things to come**

Suppose $S : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation whose standard matrix is $B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(a) Without doing any calculation, how can you tell 0 is an eigenvalue for S?

(b) Now suppose you are given the factorisation

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

In words, compare S to T (from problem 1). What do they have in common? What is different about these linear transformations?

(c) For several choices of $\mathbf{v} \in \mathbb{R}^2$, draw $\mathbf{v}$ and $T(\mathbf{v})$ on the same plane. Can you give a description in words of what T does to a vector in $\mathbb{R}^2$?

(d) Compare S to the linear transformation whose standard matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(e) (A challenge!) Let $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Can you find a matrix Q with the property that $Q[\mathbf{x}]_{\mathcal{E}} = [\text{proj}_{\mathbf{a}}\mathbf{x}]_{\mathcal{E}}$? What does this have to do with the other parts of this question?

(3) **Some other examples**

We know that eigenvectors corresponding to distinct eigenvalues are linearly independent. This question explores what happens when we have eigenvalues with multiplicity greater than one.

(a)

(b) Suppose $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation whose standard matrix is $C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.
Show that C is not diagonalisable.

(c) Suppose $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation whose standard matrix is $D = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.
Show that D is not diagonalisable.

(d) Can you find a matrix E whose characteristic equation is

$$0 = (2 - \lambda)^3$$

such that E is diagonalisable? (Don't work too hard!) Compare your matrix to D.

(e) Try to rephrase each of the parts above as a statement about the dimension of some eigenspace.