MATH1014

Semester 2

Administrative Overview

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Assessment

- Midsemester exam (date TBA) (25%)
- Final exam (45%)
- Web Assign quizzes (10%)
- Tutorial quizzes (10%)
- Tutorial participation (5%)
- Written assignment (5%)

Tips for success:

- Ask questions!
- Make use of the available resources!
- Don't fall behind!

Linear Algebra

- We will be covering most of the material in Stewart, Sections 10.1, 10.2, 10.3 and 10.4, and Lay Chapters 4 and 5, and Chapter 6, Sections 1 - 6.
- Vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$, dot products, cross products in $\mathbf{R}^3$, planes and lines in $\mathbf{R}^3$ (Stewart).
- Properties of Vector Spaces and Subspaces.
- Linear Independence, bases and dimension, change of basis.
- Applications to difference equations, Markov chains.
- Eigenvalues and eigenvectors.
- Orthogonality, Gram-Schmidt process. Least squares problem.

Coordinates, Vectors and Geometry in $\mathbb{R}^3$

From Stewart, §10.1, §10.2

Question: How do we describe 3-dimensional space?

1. Coordinates
2. Lines, planes, and spheres in $\mathbb{R}^3$
3. Vectors

Euclidean Space and Coordinate Systems

We identify points in the plane ($\mathbb{R}^2$) and in three-dimensional space ($\mathbb{R}^3$) using coordinates.

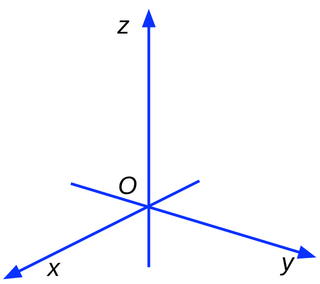
$$\mathbb{R}^3 = \{(x, y, z) \ : \ x, y, z \in \mathbb{R}\}$$

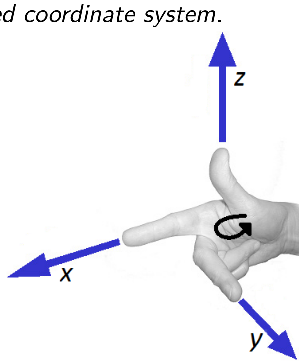
reads as "$\mathbb{R}^3$ is the set of ordered triples of real numbers".

We first choose a fixed point $\mathbf{O} = (0, 0, 0)$, called the *origin*, and three directed lines through O that are perpendicular to each other. We call these the *coordinate axes* and label them the x-axis, the y-axis and the z-axis.

Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical.

Together, $\{x, y, z\}$ form a *right-handed coordinate system*.





Compare this to the axes we use to describe $\mathbb{R}^2$, where the x-axis is horizontal and the y-axis is vertical.

The Distance Formula

Definition

The *distance* $| P_1P_2 |$ between the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$| P_1P_2 | = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Definition

The *distance* $| P_1P_2 |$ between the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is

$$| P_1P_2 | = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1.1 Surfaces in $\mathbb{R}^3$

Lines, planes, and spheres are special sets of points in $\mathbb{R}^3$ which can be described using coordinates.

Example 1

The sphere of radius r with centre $C = (c_1, c_2, c_3)$ is the set of all points in $\mathbb{R}^3$ with distance r from C:

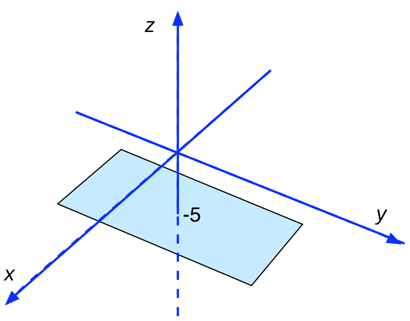
$$S = \{P \ : \ |PC| = r\}.$$

Equivalently, the sphere consists of all the solutions to this equation:

$$(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = r^2.$$

Example 2

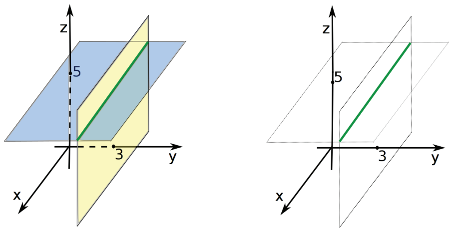
The equation $z = -5$ in $\mathbb{R}^3$ represents the set $\{(x, y, z) \mid z = -5\}$, which is the set of all points whose z-coordinate is -5. This is a horizontal plane that is parallel to the xy-plane and five units below it.



Example 3

What does the pair of equations $y = 3, z = 5$ represent? In other words, describe the set of points

$$\{(x, y, z)\ : y = 3 \text{ and } z = 5\} = \{(x, 3, 5)\}.$$



Connections with linear equations

Recall from 1013 that a system of linear equations defines a *solution set*. When we think about the unknowns as coordinate variables, we can ask what the solution set looks like.

- A single linear equation with 3 unknowns will **usually** have a solution set that's a plane. (e.g., Example 2 or $3x + 2y - 5z = 1$)
- Two linear equations with 3 unknowns will **usually** have a solution set that's a line. (e.g., Example 3 or $3x + 2y - 5z = 1$ and $x + z = 2$)
- Three linear equations with 3 unknowns will **usually** have a solution set that's a point (i.e., a unique solution).

Question

When do these heuristic guidelines fail?

Vectors

We'll study vectors both as formal mathematical objects and as tools for modelling the physical world.

Definition

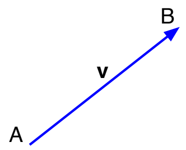
A *vector* is an object that has both magnitude and direction.

Physical quantities such as velocity, force, momentum, torque, electromagnetic field strength are all "vector quantities" in that to specify them requires both a magnitude and a direction.

Vectors

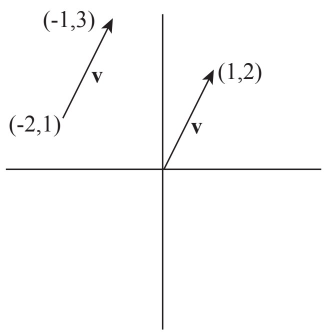
Definition

A *vector* is an object that has both magnitude and direction.



We represent vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ by arrows. For example, the vector $\mathbf{v}$ has initial point A and terminal point B and we write $\mathbf{v} = \vec{AB}$.
The *zero vector* $\mathbf{0}$ has length zero (and no direction).

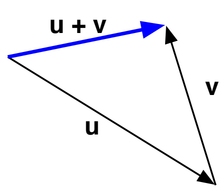
Since a vector doesn't have “location" as one of its properties, we can slide the arrow around as long as we don't rotate or stretch it.

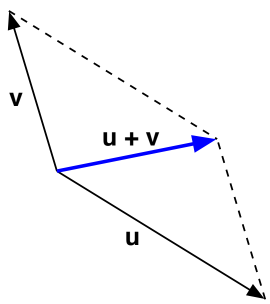


We can describe a vector using the coordinates of its head when its tail is at the origin, and we call these the *components* of the vector. Thus in this example $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and we say the components of $\mathbf{v}$ are 1 and 2.

Vector Addition

If an arrow representing $\mathbf{v}$ is placed with its tail at the head of an arrow representing $\mathbf{u}$, then an arrow from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$ represents the sum $\mathbf{u} + \mathbf{v}$.





Suppose that $\mathbf{u}$ has components a and b and that $\mathbf{v}$ has components x and y. Then $\mathbf{u} + \mathbf{v}$ has components $a + x$ and $b + y$:

$$\mathbf{u} + \mathbf{v} = (a, b) + (x, y) = (a + x, b + y)$$

Scalar Multiplication

If $\mathbf{v}$ is a vector, and t is a real number (*scalar*), then the *scalar multiple* of $\mathbf{v}$ is a vector with magnitude $|t|$ times that of $\mathbf{v}$, and direction the same as $\mathbf{v}$ if $t > 0$, or opposite to that of $\mathbf{v}$ if $t < 0$.

If $t = 0$, then $t\mathbf{v}$ is the zero vector $\mathbf{0}$.
If $\mathbf{u}$ has components a and b, then $t\mathbf{v}$ has components tx and ty:

$$t\mathbf{v} = t\langle x, y\rangle = \langle tx, ty\rangle.$$

Example

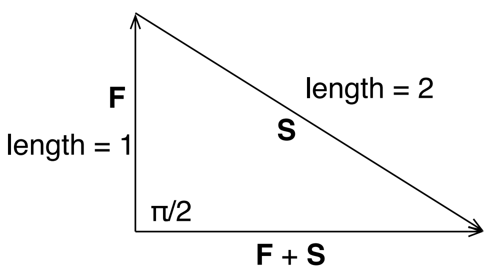
Example 4

A river flows north at 1km/hr, and a swimmer moves at 2km/hr relative to the water.

- At what angle to the bank must the swimmer move to swim east across the river?
- What is the speed of the swimmer relative to the land?

There are several velocities to be considered:
The velocity of the river, $\mathbf{F}$, with $\|\mathbf{F}\| = \mathbf{1}$;
The velocity of the swimmer relative to the water, $\mathbf{S}$, so that $\|\mathbf{S}\| = \mathbf{2}$;
The resultant velocity of the swimmer, $\mathbf{F} + \mathbf{S}$, which is to be perpendicular to $\mathbf{F}$.

The problem is to determine the *direction* of $\mathbf{S}$ and the *magnitude* of $\mathbf{F} + \mathbf{S}$.



From the figure it follows that the angle between $\mathbf{S}$ and $\mathbf{F}$ must be $2\pi/3$ and the resulting speed will be $\sqrt{3}$ km/hour. □

Standard basis vectors in $\mathbb{R}^2$

The vector **i** has components 1 and 0, and the vector **j** has components 0 and 1.

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The vector **r** from the origin to the point (x, y) has components x and y and can be expressed in the form

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{i} + y\mathbf{j}.$$

The length of of a vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is given by

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2}$$

Standard basis vectors in $\mathbb{R}^3$

In the Cartesian coordinate system in 3-space we define three **standard basis vectors i, j** and **k** represented by arrows from the origin to the points $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ respectively:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Any vector can be written as a sum of scalar multiples of the standard basis vectors:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}.$$

If $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the *length* of **v** is defined as

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}\ .$$

This is just the distance from the origin (with coordinates $0, 0, 0$) of the point with coordinates a, b, c.

A vector with length 1 is called a *unit vector*.

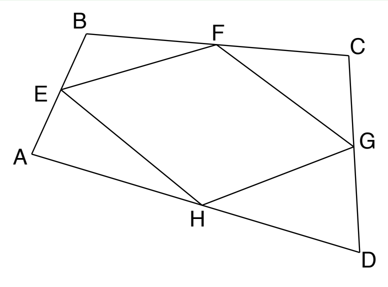
If **v** is not zero, then $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is the unit vector in the same direction as **v**.

The zero vector is not given a direction.

Vectors and Shapes

Example 5

The midpoints of the four sides of any quadrilateral are the vertices of a parallelogram.

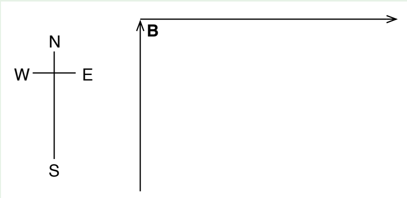


Can you prove this using vectors?

Hint: how can you tell if two vectors are parallel? How can you tell if they have the same length?

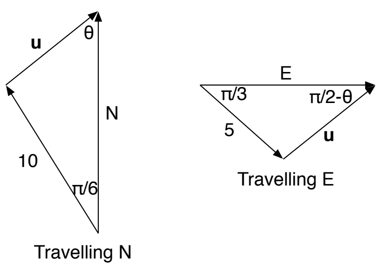
Example 6

A boat travels due north to a marker, then due east, as shown:



Travelling at a speed of 10 knots with respect to the water, the boat must head 30° west of north on the first leg because of the water current. After rounding the marker and reducing speed to 5 knots with respect to the water, the boat must be steered 60° south of east to allow for the current. Determine the velocity **u** of the water current (assumed constant).

A diagram is helpful. The vector **u** represents the velocity of the river current, and has the same magnitude and direction in both diagrams.



Applying the sine rule, we have

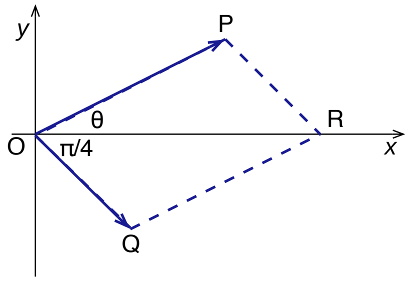
$$\frac{\sin\theta}{10} = \frac{\sin\frac{\pi}{6}}{\|\mathbf{u}\|} \qquad \frac{\cos\theta}{5} = \frac{\sin\frac{\pi}{3}}{\|\mathbf{u}\|}.$$

which are easily solvable for $\|\mathbf{u}\|$ and θ, and hence give **u**. □

Example 7

An aircraft flies with an airspeed of 750 km/h. In what direction should it head in order to make progress in a true easterly direction if the wind is from the northwest at 100 km/h?

Solution The problem is 2-dimensional, so we can use plane vectors. Choose a coordinate system so that the x- and y-axes point east and north respectively.



$$\begin{aligned}\overrightarrow{OQ} &= \mathbf{v}_{air\ \ rel\ \ ground}\\ &= 100\cos(-\pi/4)\mathbf{i} + 100\sin(-\pi/4)\mathbf{j}\\ &= 50\sqrt{2}\mathbf{i} - 50\sqrt{2}\mathbf{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{OP} &= \mathbf{v}_{aircraft\ \ rel\ \ air}\\ &= 750\cos\theta\mathbf{i} + 750\sin\theta\mathbf{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{OR} &= \mathbf{v}_{aircraft\ \ rel\ \ ground}\\ &= \overrightarrow{OP} + \overrightarrow{OQ}\\ &= (750\cos\theta\mathbf{i} + 750\sin\theta\mathbf{j}) + (50\sqrt{2}\mathbf{i} - 50\sqrt{2}\mathbf{j})\\ &= (750\cos\theta + 50\sqrt{2})\mathbf{i} + (750\sin\theta - 50\sqrt{2})\mathbf{j}\end{aligned}$$

We want $\mathbf{v}_{aircraft\ \ rel\ \ ground}$ to be in an easterly direction, that is, in the positive direction of the x-axis. So for ground speed of the aircraft v, we have

$$\overrightarrow{OR} = v\mathbf{i}.$$

Comparing the two expressions for $\overrightarrow{OR}$ we get

$$v\mathbf{i} = (750\cos\theta + 50\sqrt{2})\mathbf{i} + (750\sin\theta - 50\sqrt{2})\mathbf{j}.$$

This implies that

$$750\sin\theta - 50\sqrt{2} = 0 \quad \leftrightarrow \quad \sin\theta = \frac{\sqrt{2}}{15}.$$

This gives $\theta \approx 0.1$ radians $\approx 5.4°$.
Using this information v can be calculated, as well as the time to travel a given distance.

Overview

Last time, we used coordinate axes to describe points in space and we introduced vectors. We saw that vectors can be added to each other or multiplied by scalars.

Question: Can two vectors be multiplied?

- dot product
- cross product

(From Stewart, §10.3, §10.4)

The dot product

The *dot* or *scalar product* of two vectors is a scalar:

Definition

Given $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, the dot product of $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} &= \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{aligned}$$

Example 1

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$, then

$$\mathbf{u} \cdot \mathbf{v} = (1)(-4) + (4)(5) + (-2)(-1) = 18.$$

The following properties come directly from the definition:

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$, $k \in \mathbb{R}$

Magnitude and the dot product

Recall that if $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the *length* (or *magnitude*) of $\mathbf{v}$ is defined as

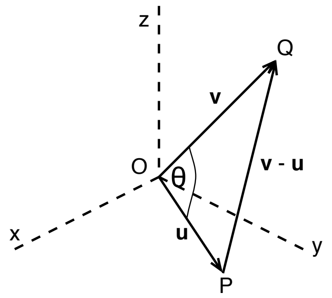
$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}\ .$$

The dot product is a convenient way to compute length:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$$

Direction and the dot product

The dot product $\mathbf{u} \cdot \mathbf{v}$ is useful for determining the relative directions of $\mathbf{u}$ and $\mathbf{v}$.
Suppose $\mathbf{u} = \overrightarrow{OP}, \mathbf{v} = \overrightarrow{OQ}$. The *angle* θ between $\mathbf{u}$ and $\mathbf{v}$ is the angle at O in the triangle POQ.



Necessarily $\theta \in [0, \pi]$.

Calculating:

$$\begin{aligned} \|\overrightarrow{PQ}\|^2 &= (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) \\ &= \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}\,. \end{aligned}$$

But the cosine rule, applied to triangle POQ, gives

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$$

whence

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta \qquad (1)$$

If either $\mathbf{u}$ or $\mathbf{v}$ are zero then the angle betwen them is not defined. In this case, however, (1) still holds in the sense that both sides are zero.

Theorem

If θ is the angle between the directions of $\mathbf{u}$ *and* $\mathbf{v}$ $(0 \leq \theta \leq \pi)$*, then*

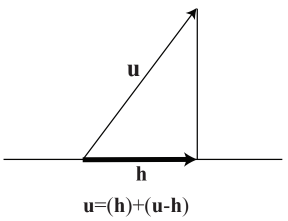
$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$$

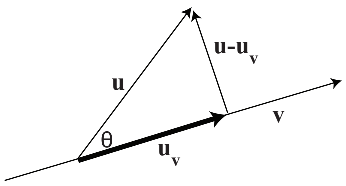
Definition

Two vectors are called *orthogonal* or *perpendicular* or *normal* if $\mathbf{u}\cdot\mathbf{v} = 0$, that is, $\theta = \pi/2$.

Scalar and vector projections

Just as we can write a vector in $\mathbb{R}^2$ as a sum of its horizontal and vertical components, we can write any vector as a sum of piece parallel to and perpendicular to a fixed vector.





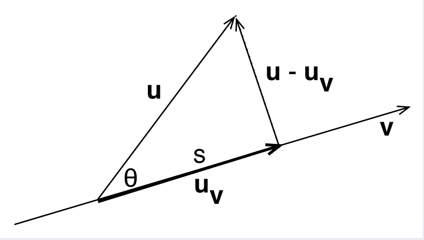
Scalar and vector projections

Definition

The *scalar projection* $s = \text{comp}_{\mathbf{v}}\mathbf{u}$ of any vector $\mathbf{u}$ in the direction of the nonzero vector $\mathbf{v}$ is the scalar product of $\mathbf{u}$ with a unit vector in the direction of $\mathbf{v}$.

$$\text{comp}_{\mathbf{v}}\mathbf{u} = \mathbf{u}\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\| \cos\theta$$

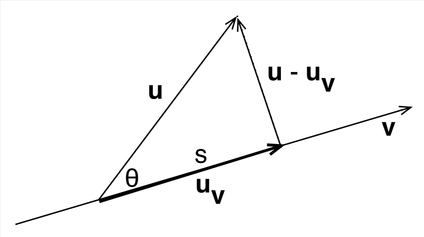
where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.



Definition

The *vector projection* $\mathbf{u}_\mathbf{v} = \text{proj}_\mathbf{v}\mathbf{u}$ of $\mathbf{u}$ in the direction of the nonzero vector $\mathbf{v}$ is the scalar multiple of a unit vector $\hat{\mathbf{v}}$ in the direction of $\mathbf{v}$, by the scalar projection of $\mathbf{u}$ in the direction $\mathbf{v}$:

$$\text{proj}_\mathbf{v}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\hat{\mathbf{v}} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}.$$



In words:

- The scalar projection of $\mathbf{u}$ onto $\mathbf{v}$ is...
- The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is...

Remember that we can write $\mathbf{u}$ as a sum of a vector parallel to $\mathbf{v}$ and a vector perpendicular to $\mathbf{v}$. We call the summand parallel to $\mathbf{v}$ the *component* in the $\mathbf{v}$ direction.

- The scalar projection of $\mathbf{u}$ onto $\mathbf{v}$ is the length of the component of $\mathbf{u}$ in the $\mathbf{v}$ direction.
- The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is the component of $\mathbf{u}$ in the $\mathbf{v}$ direction.

Definition of the cross product

In $\mathbb{R}^3$ *only*, there is a product of two vectors called a *cross product* or *vector product*. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is a vector denoted $\mathbf{a}\times\mathbf{b}$.

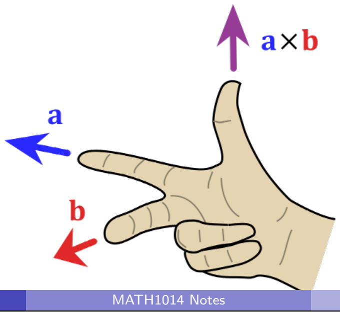
To specify a vector in $\mathbb{R}^3$, we need to give its magnitude and direction.

Definition of the cross product

Definition

Given **a** and **b** in $\mathbb{R}^3$ with $\theta \in [0, \pi]$ the angle between them, the cross product $\mathbf{a} \times \mathbf{b}$ is the vector defined by the following properties:

- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$
- $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**
- $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ form a right-handed coordinate system



Computing cross products

Given $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, how can we find the coordinates of $\mathbf{a} \times \mathbf{b}$?

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

You should check that this formula gives a vector satisfying the definition on the previous slide! Alternatively, we could give this formula as the definition and then prove those properties as a theorem.

In order to make the definition easier to remember we use the notation of determinants. Recall that a **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Further a **determinant of order 3** can be defined in terms of second order determinants:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

We now rewrite the cross product using determinants of order 3 and the standard basis vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

In view of the similarity of the last two equations we often write

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{2}$$

Although the first row of the symbolic determinant in Equation 2 consists of vectors, it can be expanded as if it were an ordinary determinant.

Example 2

Find a vector with positive $\mathbf{k}$ component which is perpendicular to both $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Solution The vector $\mathbf{a} \times \mathbf{b}$ will be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} \\ &= -7\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}. \end{aligned}$$

Now we require a vector with a positive $\mathbf{k}$. It is given by $\langle 7, 6, 4 \rangle$.

Properties of the cross product

Lemma

Two non zero vectors $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel (or antiparallel) if and only if*

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

Properties of the cross product

If **u** **v** and **w** are any vectors in $\mathbb{R}^3$, and t is a real number, then

1. $\mathbf{u}\times\mathbf{v} = - \ldots.$
2. $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \ldots.$
3. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \ldots.$
4. $(t\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(t\mathbf{v}) = \ldots.$
5. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \ldots.$
6. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \ldots$

Properties of the cross product

If **u** **v** and **w** are any vectors in $\mathbb{R}^3$, and t is a real number, then...

1. $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}.$
2. $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}.$
3. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}.$
4. $(t\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(t\mathbf{v}) = t(\mathbf{u}\times\mathbf{v}).$
5. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}.$
6. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$

Note the absence of an associative law. The cross product is not associative. In general

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \neq (\mathbf{u}\times\mathbf{v})\times\mathbf{w}!$$

Comparing the dot and cross product

- Where is each defined?
- What is the output?
- What's the significance of zero?
- Is it commutative?

Example 3

A triangle ABC has vertices $(2,-1,0)$, $(5,-4,3)$, $(1,-3,2)$. Is it a right triangle?

The sides are $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$, $\overrightarrow{AC} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$, $\overrightarrow{BC} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$.

Since

$$\cos\theta_C = \frac{\overrightarrow{AC}\cdot\overrightarrow{BC}}{\|\overrightarrow{AC}\|\|\overrightarrow{BC}\|} = \frac{(-1)(-4)+(-2)(1)+(2)(-1)}{\|\overrightarrow{AC}\|\|\overrightarrow{BC}\|} = \frac{0}{\|\overrightarrow{AC}\|\|\overrightarrow{BC}\|} = 0,$$

the sides $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are orthogonal.

Example 4

For what value of k do the four points $A = (1,1,-1)$, $B = (0,3,-2)$, $C = (-2,1,0)$ and $D = (k,0,2)$ all lie in a plane?

Solution The points A, B and C form a triangle and all lie in the plane containing this triangle. We need to find the value of k so that D is in the same plane.

One way of doing this is to find a vector $\mathbf{u}$ perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and then find k so that $\overrightarrow{AD}$ is perpendicular to $\mathbf{u}$.

A suitable vector $\mathbf{u}$ is given by $\overrightarrow{AB}\times\overrightarrow{AC}$. We then require that

$$\mathbf{u}\cdot\overrightarrow{AD} = 0.$$

Putting this together we require that

$$(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD} = \mathbf{0}.$$

Example (continued)

For what value of k do the four points $A = (1,1,-1)$, $B = (0,3,-2)$, $C = (-2,1,0)$ and $D = (k,0,2)$ all lie in a plane?

Now

$$\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \overrightarrow{AC} = -3\mathbf{i} + \mathbf{k}, \quad \overrightarrow{AD} = (k-1)\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

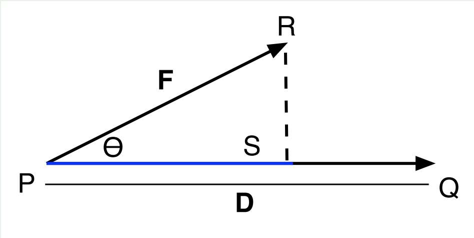
Then

$$\begin{aligned}
(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD} &= \overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) \\
&= \begin{vmatrix} k-1 & -1 & 3 \\ -1 & 2 & -1 \\ -3 & 0 & 1 \end{vmatrix} \\
&= (k-1)2 - (-1)(-4) + 3(6) \\
&= 2k - 2 - 4 + 18 \\
&= 2k + 12
\end{aligned}$$

So $(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD} = 0$ when $k = -6$, and D lies on the required plane

Example 5

One use of projections occurs in physics in calculating work.



Suppose a constant force $\mathbf{F} = \vec{PR}$ moves an object from P to Q. The **displacement vector** is $\mathbf{D} = \vec{PQ}$. The **work** done by this force is defined to be the product of the component of the force along $\mathbf{D}$ and the distance moved:

$$W = (\|\mathbf{F}\| \cos\theta)\, \|\mathbf{D}\| = \mathbf{F} \cdot \mathbf{D}.$$

Example 6

Let $\mathbf{a} = \langle 1, 3, 0\rangle$ and $\mathbf{b} = \langle -2, 0, 6\rangle$, Then

$$\begin{aligned}
\text{comp}_{\mathbf{a}}\mathbf{b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} \\
&= \frac{-2+0+0}{\sqrt{1+9+0}} = \frac{-2}{\sqrt{10}}. \\
\text{proj}_{\mathbf{a}}\mathbf{b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\hat{\mathbf{a}} \\
&= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} \\
&= \frac{-2}{\sqrt{10}}\frac{\langle 1,3,0\rangle}{\sqrt{10}} \\
&= \frac{\langle -2,-6,0\rangle}{10} = \langle -1/5, -3/5, 0\rangle.
\end{aligned}$$

Overview

Last week we introduced vectors in Euclidean space and the operations of vector addition, scalar multiplication, dot product, and (for $\mathbb{R}^3$) cross product.

Question

How can we use vectors to describe lines and planes in $\mathbb{R}^3$?

(From Stewart §10.5)

Warm-up

Question

Describe all the vectors in $\mathbb{R}^3$ which are orthogonal to the 0 vector. Can you rephrase your answer as a statement about solutions to some linear equation?

Remember that the statement "$\mathbf{v}$ is orthogonal to $\mathbf{u}$" is equivalent to "$\mathbf{v} \cdot \mathbf{u} = 0$".

This question asks for all the vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.$

Using the definition of the dot product, this translates to asking what $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfy the equation $0x + 0y + 0z = 0$...

...the answer is that all vectors in $\mathbb{R}^3$ are orthogonal to the 0 vector. Equivalently, every triple (x, y, z) is a solution to the linear equation $0x + 0y + 0z = 0$.

Lines in $\mathbb{R}^2$

In the xy-plane the general form of the equation of a line is

$$ax + by = c,$$

where a and b are not both zero. If $b \neq 0$ then this equation can be rewritten as

$$y = -(a/b)x + c/b,$$

which has the form $y = mx + k$. (Here m is the slope of the line and the point $(0, k)$ is its y-intercept.)

Example 1

Let L be the line $2x + y = 3$. The line has slope $m = -2$ and the y-intercept is $(0, 3)$.

Alternatively, we could think about this line ($y = -2x + 3$) as the path traced out by a moving particle.

Suppose that the particle is initially at the point $(0, 3)$ at time $t = 0$. Suppose, too, that its x-coordinate changes at a constant rate of 1 unit per second and its y-coordinate changes as a constant rate of -2 units per second.

At $t = 1$ the particle is at $(1, 1)$. If we assume it's always been moving this way, then we also know that at $t = -2$ it was at $(-2, 7)$. In general, we can display the relationship in vector form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ -2t + 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

What is the significance of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$?

In this expression, $\mathbf{v}$ is a vector parallel to the line L, and is called a *direction vector* for L. The previous example shows that we can express L in terms of a direction vector and a vector to specific point on L:

Definition

The equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

is the *vector equation* of the line L. The variable t is called a *parameter*.

Here, $\mathbf{r}_0$ is the vector to a specific point on L; any vector $\mathbf{r}$ which satisfies this equation is a vector to some point on L.

Example 2

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad (1)$$

is the *vector equation* of the line L.

If we express the vectors in a vector equation for L in components, we get a collection of equations relating scalars.

Definition

For $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, the *parametric equations* of the line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ are

$$x = x_0 + ta$$
$$y = y_0 + tb.$$

Lines in $\mathbb{R}^3$

The definitions of the vector and parametric forms of a line carry over perfectly to $\mathbb{R}^3$.

Definition

The *vector form of the equation of the line* L in $\mathbb{R}^2$ or $\mathbb{R}^3$ is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}_0$ is a specific point on L and $\mathbf{v} \neq \mathbf{0}$ is a direction vector for L. The equations corresponding to the components of the vector form of the equation are called *parametric equations* of L.

Example 3

Let $\mathbf{r}_0 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Then the vector equation of the line L is

$$\mathbf{r} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

The line L contains the point $(1, 4, -2)$ and has direction parallel to $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. By taking different values of t we can find different points on the line.

Question

For a given line, is the vector equation for the line unique?

No, any vector parallel to the direction vector is another direction vector, and each choice of a point on L will give a different $\mathbf{r}_0$.

Example 4

The line with parametric equations

$$x = 1 + 2t \quad y = -4t \quad z = -3 + 5t.$$

can also be expressed as

$$x = 3 + 2t \quad y = -4 - 4t \quad z = 2 + 5t.$$

or as

$$x = 1 - 4t \quad y = 8t \quad z = -3 - 10t.$$

Note that a fixed value of t corresponds to three different points on L when plugged into the three different systems.

Symmetric equations of a line

Another way of describing a line L is to eliminate the parameter t from the parametric equations

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

If $a \neq 0$, $b \neq 0$ and $c \neq 0$ then we can solve each of the scalar equations for t and obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

These equations are called the *symmetric equations* of the line L through (x_0, y_0, z_0) parallel to $\mathbf{v}$. The numbers a, b and c are called the *direction numbers* of L.
If, for example $a = 0$, the equation becomes

$$x = x_0, \ \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example 5

Find parametric and symmetric equations for the line through $(1, 2, 3)$ and parallel to $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.

The line has the vector parametric form

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}),$$

or scalar parametric equations

$$\begin{cases} x = 1 + 2t \\ y = 2 + 3t \\ z = 3 - 4t \end{cases} \qquad (-\infty < t < \infty).$$

Its symmetric equations are

$$\frac{x - 1}{2} = \frac{y - 2}{3} = \frac{z - 3}{-4}.$$

Example 6

Determine whether the two lines given by the parametric equations below intersect

$$L_1 : x = 1 + 2t, y = 3t, z = 2 - t$$

$$L_2 : x = -1 + s, y = 4 + s, z = 1 + 3s$$

If L_1 and L_2 intersect, there will be values of s and t satisfying

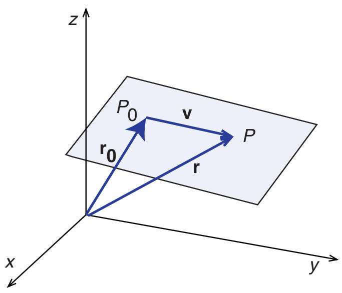
$$\begin{aligned} 1 + 2t &= -1 + s \\ 3t &= 4 + s \\ 2 - t &= 1 + 3s \end{aligned}$$

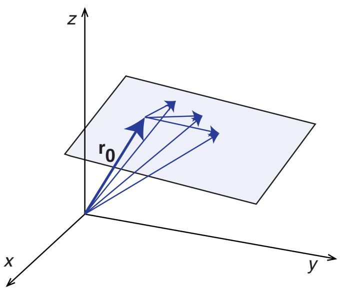
Solving the first two equations gives $s = 14, t = 6$, but these values don't satisfy the third equation. We conclude that the lines L_1 and L_2 don't intersect.
In fact, their direction vectors are not proportional, so the lines aren't parallel, either. They are *skew* lines.

Planes in $\mathbb{R}^3$

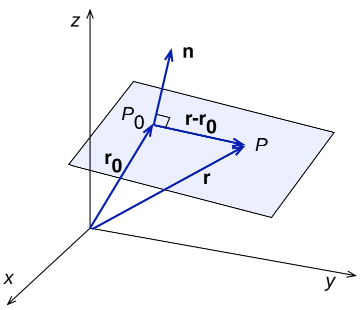
We described a line as the set of position vectors expressible as $\mathbf{r}_0 + \mathbf{v}$, where $\mathbf{r}_0$ was a position vector of a point in L and $\mathbf{v}$ was any vector parallel to L.

We can describe a plane the same way: the set of position vectors expressible as the sum of a position vector to a point in P and an arbitrary vector parallel to P.





Choose a vector $\mathbf{n}$ which is orthogonal to the plane and choose an arbitrary point P_0 in the plane.



How can we use this data to describe all the other points P which lie in the plane?
Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P respectively.
The normal vector $\mathbf{n}$ is orthogonal to every vector in the plane. In particular $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

This equation

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0. \tag{2}$$

can be rewritten as

$$\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{r}_0. \tag{3}$$

Either of the equations (2) or (3) is called a *vector equation of the plane*.

Example 7

Find a vector equation for the plane passing through $P_0 = (0, -2, 3)$ and normal to the vector $\mathbf{n} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

We have $\mathbf{r}_0 = \langle 0, -2, 3\rangle$ and $\mathbf{n} = \langle 4, 2, -3\rangle$. Thus the vector form is

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0,$$

or

$$(4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})\cdot[(x - 0)\mathbf{i} + (y + 2)\mathbf{j} + (z - 3)\mathbf{k}] = 0.$$

Expanding this gives us a *scalar equation* for the plane...

Given $\mathbf{n} = \langle A, B, C\rangle$, $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$, the vector equation $\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0$ becomes

$$\langle A, B, C\rangle\cdot\langle x - x_0, y - y_0, z - z_0\rangle = 0,$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{4}$$

Equation (4) is the *scalar equation of the plane* through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle A, B, C\rangle$.

The equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

can be written more simply in **standard form**

$$Ax + By + Cz + D = 0,$$

where $D = -(Ax_0 + By_0 + Cz_0)$.

If $D = 0$, the plane passes through the origin.

Example 8

Find a scalar equation for the plane passing through $P_0 = (0, -2, 3)$ and normal to the vector $\mathbf{n} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

The vector form is

$$(4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot [(x - 0)\mathbf{i} + (y + 2)\mathbf{j} + (z - 3)\mathbf{k}] = 0,$$

which in scalar form becomes

$$4(x - 0) + 2(y + 2) - 3(z - 3) = 0$$

and this is equivalent to

$$4x + 2y - 3z = -13.$$

Example 9

Find a scalar equation of the plane containing the points

$$P = (1, 1, 2), \quad Q = (0, 2, 3), \quad R = (-1, -1, -4).$$

First, we should find a normal vector $\mathbf{n}$ to the plane, and there are several ways to do this.
The vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ will be perpendicular to $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{PR} = -2\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$. Therefore, we can solve a system of linear equations:

$$0 = \mathbf{n} \cdot (-\mathbf{i} + \mathbf{j} + \mathbf{k}) = -n_1 + n_2 + n_3$$

$$0 = \mathbf{n} \cdot (-2\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}) = -2n_1 - 2n_2 - 6n_3.$$

One solution to this system is $\mathbf{n} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, so this is an example of a normal vector to the plane containing the 3 given points.

We can use this normal vector $\mathbf{n} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, together with any one of the given points to write the equation of the plane. Using $Q = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, the equation is

$$-(x - 0) - 2(y - 2) + 1(z - 3) = 0,$$

which simplifies to

$$x + 2y - z = 1.$$

The first step in this example was finding the normal vector **n**, but in fact, there's another way to do this.

Recall that *in* $\mathbb{R}^3$ *only*, there is a product of two vectors called a *cross product*. The cross product of **a** and **b** is a vector denoted $\mathbf{a} \times \mathbf{b}$ which is orthogonal to both **a** and **b**. If we have two nonzero vectors **a** and **b** parallel to our plane, then $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is a normal vector.

Example 10

Consider the two planes

$$x - y + z = -1 \quad \text{and} \quad 2x + y + 3z = 4.$$

Explain why the planes above are not parallel and find a direction vector for the line of intersection.

Two planes are parallel if and only if their normal vectors are parallel. Normal vectors for the two planes above are for example

$$\mathbf{n}_1 = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

respectively. These vectors are not parallel, so the planes can't be parallel and must intersect. A vector **v** parallel to the line of intersection is a vector which is orthogonal to both the normal vectors above. We can find such a vector by calculating the cross product of the normal vectors:

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -4\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

Example 11

Find the line through the origin and parallel to the line of intersection of the two planes

$$x + 2y - z = 2 \quad \text{and} \quad 2x - y + 4z = 5.$$

The planes have respective normals

$$\mathbf{n}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}.$$

A direction vector for their line of intersection is given by

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = 7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.$$

A vector parametric equation of the line is

$$\mathbf{r} = t(7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}),$$

since the line passes through the origin.

Parametric equations for this line are, for example,

$$\begin{aligned} x &= 7t \\ y &= -6t \\ z &= -5t \end{aligned}$$

and the corresponding symmetric equations are

$$\frac{x}{7} = \frac{y}{-6} = \frac{z}{-5}.$$

Recommended exercises for review

Stewart §10.5: 1, 3, 15, 19, 25, 29, 35

Overview

Yesterday we introduced equations to describe lines and planes in $\mathbb{R}^3$:

- $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$
  The vector equation for a line describes arbitrary points $\mathbf{r}$ in terms of a specific point $\mathbf{r_0}$ and the direction vector $\mathbf{v}$.
- $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$
  The vector equation for a plane describes arbitrary points $\mathbf{r}$ in terms of a specific point $\mathbf{r_0}$ and the normal vector $\mathbf{n}$.

Question

How can we find the distance between a point and a plane in $\mathbb{R}^3$? Between two lines in $\mathbb{R}^3$? Between two planes? Between a plane and a line?

(From Stewart §10.5)

Distances in $\mathbb{R}^3$

The distance between two points is the length of the line segment connecting them. However, there's more than one line segment from a point P to a line L, so what do we mean by the *distance* between them?

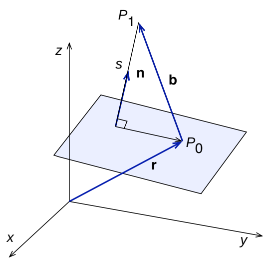
The distance between any two subsets A, B of $\mathbb{R}^3$ is the smallest distance between points a and b, where a is in A and b is in B.

- To determine the distance between a point P and a line L, we need to find the point Q on L which is closest to P, and then measure the length of the line segment PQ.
  This line segment is *orthogonal* to L.
- To determine the distance between a point P and a plane S, we need to find the point Q on S which is closest to P, and then measure the length of the line segment PQ.
  Again, this line segment is *orthogonal* to S.

In both cases, the key to computing these distances is drawing a picture and using one of the vector product identitites.

Distance from a point to a plane

We find a formula for the distance s from a point $P_1 = (x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$.



Let $P_0 = (x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b}$ be the vector corresponding to $\vec{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle.$$

The distance s from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle A, B, C \rangle$.

$$\begin{aligned} s &= |\,\text{comp}_{\mathbf{n}}\mathbf{b}\,| \\ &= \frac{|\,\mathbf{n}\cdot\mathbf{b}\,|}{||\mathbf{n}||} \\ &= \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|Ax_1 + By_1 + Cz_1 - (Ax_0 + By_0 + Cz_0)|}{\sqrt{A^2 + B^2 + C^2}} \end{aligned}$$

Since P_0 is on the plane, its coordinates satisfy the equation of the plane and so we have $Ax_0 + By_0 + Cz_0 + D = 0$. Thus the formula for s can be written

$$s = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Example 1

We find the distance from the point $(1, 2, 0)$ to the plane $3x - 4y - 5z - 2 = 0$.

From the result above, the distance s is given by

$$s = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

where $(x_0, y_0, z_0) = (1, 2, 0)$,

$$A = 3, B = -4, C = -5 \text{ and } D = -2.$$

This gives

$$\begin{aligned} s &= \frac{|3 \cdot 1 + (-4) \cdot 2 + (-5) \cdot 0 - 2|}{\sqrt{3^2 + (-4)^2 + (-5)^2}} \\ &= \frac{7}{\sqrt{50}} = \frac{7}{5\sqrt{2}} = \frac{7\sqrt{2}}{10}. \end{aligned}$$

Distance from a point to a line

Question

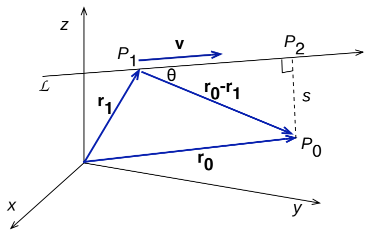
Given a point $P_0 = (x_0, y_0, z_0)$ and a line L in $\mathbb{R}^3$, what is the distance from P_0 to L?

Tools:

- describe L using vectors
- $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||\,||\mathbf{v}|| \sin\theta$

Distance from a point to a line

Let $P_0 = (x_0, y_0, z_0)$ and let L be the line through P_1 and parallel to the nonzero vector $\mathbf{v}$. Let $\mathbf{r}_0$ and $\mathbf{r}_1$ be the position vectors of P_0 and P_1 respectively. P_2 on L is the point closest to P_0 if and only if the vector $\overrightarrow{P_2P_0}$ is perpendicular to L.



The distance from P_0 to L is given by

$$s = ||\overrightarrow{P_2P_0}|| = ||\overrightarrow{P_1P_0}|| \sin\theta = ||\mathbf{r}_0 - \mathbf{r}_1|| \sin\theta$$

where θ is the angle between $\mathbf{r}_0 - \mathbf{r}_1$ and $\mathbf{v}$

Since

$$||(\mathbf{r}_0 - \mathbf{r}_1) \times \mathbf{v}|| = ||\mathbf{r}_0 - \mathbf{r}_1|| \, ||\mathbf{v}|| \sin\theta$$

we get the formula

$$\begin{aligned} s &= ||\mathbf{r}_0 - \mathbf{r}_1|| \sin\theta \\ &= \frac{||(\mathbf{r}_0 - \mathbf{r}_1) \times \mathbf{v}||}{||\mathbf{v}||} \end{aligned}$$

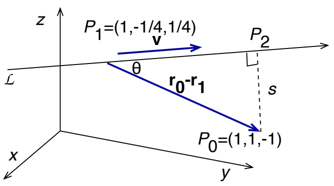
Example 2

Find the distance from the point $(1, 1, -1)$ to the line of intersection of the planes

$$x + y + z = 1, \quad 2x - y - 5z = 1.$$

The direction of the line is given by $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$ where $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} - 5\mathbf{k}$.

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = -4\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}.$$



In the diagram, P_1 is an arbitrary point on the line. To find such a point, put $x = 1$ in the first equation. This gives $y = -z$ which can be used in the second equation to find $z = 1/4$, and hence $y = -1/4$.

Here $\overrightarrow{P_1P_0} = \mathbf{r}_0 - \mathbf{r}_1 = \frac{5}{4}\mathbf{j} - \frac{5}{4}\mathbf{k}$. So

$$\begin{aligned} s &= \frac{||(\mathbf{r}_0 - \mathbf{r}_1)\times\mathbf{v}||}{||\mathbf{v}||} \\ &= \frac{||(\frac{5}{4}\mathbf{j} - \frac{5}{4}\mathbf{k})\times(-4\mathbf{i} + 7\mathbf{j} - 3\mathbf{k})||}{\sqrt{(-4)^2 + 7^2 + (-3)^2}} \\ &= \frac{||5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}||}{\sqrt{74}} \\ &= \sqrt{\frac{75}{74}}. \end{aligned}$$

Distance between two lines

Let L_1 and L_2 be two lines in $\mathbb{R}^3$ such that

- L_1 passes through the point P_1 and is parallel to the vector $\mathbf{v}_1$
- L_2 passes through the point P_2 and is parallel to the vector $\mathbf{v}_2$.

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of P_1 and P_2 respectively.
Then parametric equation for these lines are

$$L_1 \qquad \mathbf{r} = \mathbf{r}_1 + t\mathbf{v}_1$$

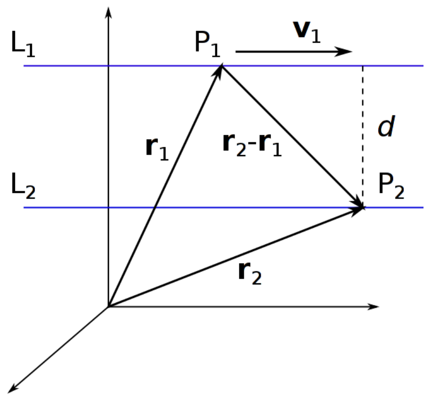
$$L_2 \qquad \tilde{\mathbf{r}} = \mathbf{r}_2 + s\mathbf{v}_2$$

Note that $\mathbf{r}_2 - \mathbf{r}_1 = \overrightarrow{P_1P_2}$.
We want to compute the smallest distance d (simply called the distance) between the two lines.
If the two lines intersect, then $d = 0$. If the two lines do not intersect we can distinguish two cases.

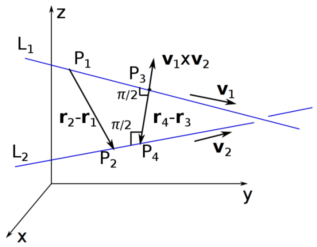
Case 1: L_1 and L_2 are parallel and do not intersect.

In this case the distance d is simply the distance from the point P_2 to the line L_1 and is given by

$$d = \frac{||\overrightarrow{P_1P_2} \times \mathbf{v}_1||}{||\mathbf{v}_1||} = \frac{||(\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{v}_1||}{||\mathbf{v}_1||}$$



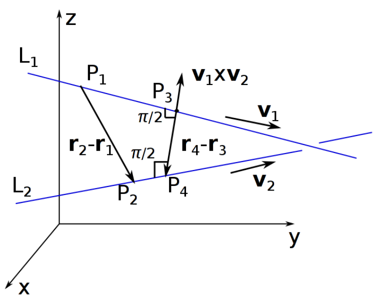
Case 2: L_1 and L_2 are skew lines.



If P_3 and P_4 (with position vectors $\mathbf{r}_3$ and $\mathbf{r}_4$ respectively) are the points on L_1 and L_2 that are closest to one another, then the vector $\overrightarrow{P_3P_4}$ is perpendicular to both lines (i.e. to both $\mathbf{v}_1$ and $\mathbf{v}_2$) and therefore parallel to $\mathbf{v}_1 \times \mathbf{v}_2$. The distance d is the length of $\overrightarrow{P_3P_4}$.
Notice that $d = ||\mathbf{r}_4 - \mathbf{r}_3||$, which we can rewrite as

$$d = \frac{|(\mathbf{r}_4 - \mathbf{r}_3) \cdot (\mathbf{v}_1 \times \mathbf{v}_2)|}{||\mathbf{v}_1 \times \mathbf{v}_2||}$$

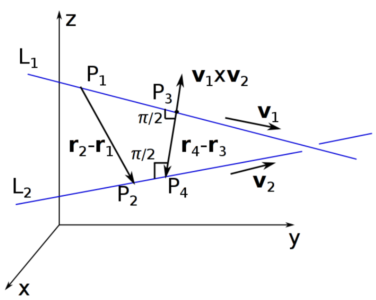
because $\mathbf{r}_4 - \mathbf{r}_3$ is parallel to $\mathbf{v}_1 \times \mathbf{v}_2$).



What's the point of doing this? Of course we don't know what $\mathbf{r}_4$ or $\mathbf{r}_3$ is. Here's the trick: Notice that

$$\mathbf{r}_4 = \mathbf{r}_2 + t\mathbf{v}_2 \qquad\qquad \mathbf{r}_3 = \mathbf{r}_1 + s\mathbf{v}_1$$

for some s and t.



Now substitute these into our dimension formula, obtaining

$$d = \frac{|(\mathbf{r}_2 - \mathbf{r}_1 + t\mathbf{v}_2 - s\mathbf{v}_1) \cdot (\mathbf{v}_1 \times \mathbf{v}_2)|}{||\mathbf{v}_1 \times \mathbf{v}_2||}$$

which simplifies, since $\mathbf{v}_1 \times \mathbf{v}_2$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_1$, to

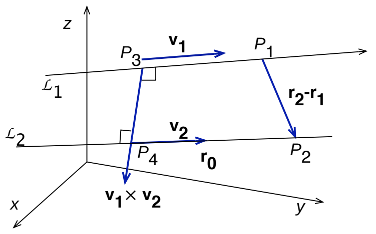
$$d = \frac{|(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{v}_1 \times \mathbf{v}_2)|}{||\mathbf{v}_1 \times \mathbf{v}_2||}$$

Thus we don't need to know $\mathbf{r}_4$ or $\mathbf{r}_3$ explicitly at all! (Exercise — find formulas for them!)

Example 3

Find the distance between the skew lines

$$\begin{cases} x + 2y = 3 \\ y + 2z = 3 \end{cases} \quad \text{and} \quad \begin{cases} x + y + z = 6 \\ x - 2z = -5 \end{cases}$$



We can take $P_1 = (1, 1, 1)$, a point on the first line, and $P_2 = (1, 2, 3)$ a point on the second line. This gives $\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{j} + 2\mathbf{k}$.

Now we need to find $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{v}_1 = (\mathbf{i} + 2\mathbf{j}) \times (\mathbf{j} + 2\mathbf{k}) = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k},$$

and

$$\mathbf{v}_2 = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - 2\mathbf{k}) = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}.$$

This gives

$$\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}.$$

The required distance d is the length of the projection of $\mathbf{r}_2 - \mathbf{r}_1$ in the direction of $\mathbf{v}_1 \times \mathbf{v}_2$, and is given by

$$\begin{aligned} d &= \frac{|(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{v}_1 \times \mathbf{v}_2)|}{||\mathbf{v}_1 \times \mathbf{v}_2||} \\ &= \frac{|(\mathbf{j} + 2\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + 8\mathbf{k})|}{\sqrt{(-1)^2 + 2^2 + 8^2}} \\ &= \frac{18}{\sqrt{69}}. \end{aligned}$$

Overview

We've studied the geometric and algebraic behaviour of vectors in Euclidean space. This week we turn to an abstract model that has many of the same algebraic properties.

The importance of this is two-fold:

- Many models of physical processes do not sit in $\mathbb{R}^3$, or indeed in $\mathbb{R}^n$ for any n.
- Apparently different situations often turn out to be "essentially" the same; studying the abstract case solves many problems at once.

(Lay, §4.1)

Let's review vector operations in language that will help set up our generalisation:

- Vectors are objects which can be added together or multiplied by scalars; both operations give back a vector.
- Vector addition is commutative and associative; scalar multiplication and vector addition are distributive.
- Adding the zero vector to $\mathbf{v}$ doesn't change $\mathbf{v}$.
- Multiplying a vector $\mathbf{v}$ by the scalar 1 doesn't change $\mathbf{v}$.
- Adding $\mathbf{v}$ to $(-1)\mathbf{v}$ gives the zero vector.

(Notice that we haven't included the dot product. This does have a role to play in our abstract setting, but we'll come to it later in the term.)

Definition

A *vector space* is a non-empty set V of objects called *vectors* on which are defined operations of *addition* and *multiplication by scalars*. These objects and operations must satisfy the following ten axioms for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

For now, we'll take the set of scalars to be the real numbers. In a few weeks, we'll consider vector spaces where the scalars are complex numbers instead.

Definition

A *vector space* is a non-empty set V of objects called *vectors* on which are defined operations of *addition* and *multiplication by scalars*. These objects and operations must satisfy the following ten axioms for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

The axioms for a vector space

1. $\mathbf{u} + \mathbf{v}$ is in V;
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$; (commutativity)
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$; (associativity)
4. there is an element $\mathbf{0}$ in V, $\mathbf{0} + \mathbf{u} = \mathbf{u}$;
5. there is $-\mathbf{u} \in V$ with $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
6. $c\mathbf{u}$ is in V;
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$;
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$;
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$;
10. $1\mathbf{u} = \mathbf{u}$.

Example 1

Let $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$, with the usual operations of addition of matrices and multiplication by a scalar.

In this context the the zero vector $\mathbf{0}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The negative of the vector $\mathbf{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $-\mathbf{v} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$.

For the same vector $\mathbf{v}$ and $t \in \mathbb{R}$ we have $t\mathbf{v} = \begin{bmatrix} ta & tb \\ tc & td \end{bmatrix}$.

If $\mathbf{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ then $\mathbf{u} + \mathbf{w} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$.

Example 2

Let $\mathbb{P}_2$ be the set of all polynomials of degree at most 2 with coefficients in $\mathbb{R}$. Elements of $\mathbb{P}_2$ have the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$$

where a_0, a_1 and a_2 are real numbers and t is a real variable. You are already familiar with adding two polynomials or multiplying a polynomial by a scalar.

The set $\mathbb{P}_2$ is a vector space.

We will just verify 3 out of the 10 axioms here.

Let $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$ and $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2$, and let c be a scalar.

Axiom 1: $\mathbf{v} + \mathbf{u}$ is in V

The polynomial $\mathbf{p} + \mathbf{q}$ is defined in the usual way: $(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$. Therefore,

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$$

which is also a polynomial of degree at most 2. So $\mathbf{p} + \mathbf{q}$ is in $\mathbb{P}_2$.

Axiom 4: $\mathbf{v} + \mathbf{0} = \mathbf{v}$

The zero vector $\mathbf{0}$ is the zero polynomial $\mathbf{0} = 0 + 0t + 0t^2$.

$$(\mathbf{p} + \mathbf{0})(t) = \mathbf{p}(t) + \mathbf{0}(t) = (a_0 + 0) + (a_1 + 0)t + (a_2 + 0)t^2 = \mathbf{p}(t).$$

So $\mathbf{p} + \mathbf{0} = \mathbf{p}$.

Axiom 6: $c\mathbf{u}$ is in V

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = (ca_0) + (ca_1)t + (ca_2)t^2.$$

This is again a polynomial in $\mathbb{P}_2$.

The remaining 7 axioms also hold, so $\mathbb{P}_2$ is a vector space.

In fact, the previous example generalises:

Example 3

Let $\mathbb{P}_n$ be the set of polynomials of degree at most n with coefficients in $\mathbb{R}$. Elements of $\mathbb{P}_n$ are polynomials of the form

$$\mathbf{p}(t) = a_0 + a_1 t + \ldots + a_n t^n$$

where $a_0, a_1, \ldots, a_n$ are real numbers and t is a real variable.

As in the example above, the usual operations of addition of polynomials and multiplication of a polynomial by a real number make $\mathbb{P}_n$ a vector space.

Example 4

The set $\mathbb{Z}$ of integers with the usual operations *is not* a vector space. To demonstrate this it is enough to to find that *one* of the ten axioms fails and to give a specific instance in which it fails (i.e., a *counterexample*).

In this case we find that we do not have closure under scalar multiplication (Axiom 6). For example, the multiple of the integer 3 by the scalar $\frac{1}{4}$ is

$$\left(\frac{1}{4}\right)(3) = \frac{3}{4}$$

which is not an integer. Thus it is not true that cx is in $\mathbb{Z}$ for *every* x in $\mathbb{Z}$ and *every* scalar c.

Example 5

Let $\mathcal{F}$ denote the set of real valued functions defined on the real line. If f and g are two such functions and c is a scalar, then $f+g$ and cf are defined by

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (cf)(x) = cf(x).$$

This means that the *value* of $f+g$ at x is obtained by adding together the values of f and g at x. So if f is the function $f(x) = \cos x$ and g is $g(x) = e^x$ then

$$(f+g)(0) = f(0) + g(0) = \cos 0 + e^0 = 1 + 1 = 2.$$

We find cf in a similar way. This means axioms 1 and 6 are true. The other axioms need to be verified, and with that verification $\mathcal{F}$ is a vector space.

Sometimes we have vector spaces with *unintuitive* operations for addition and scalar multiplication.

Example 6

Consider $\mathbb{R}_{>0}$, the positive real numbers, under the following operations:

- $\mathbf{v} \oplus \mathbf{w} = \mathbf{vw}$
- $c \otimes \mathbf{v} = \mathbf{v}^c$.

Counterintuitively, this is a vector space! For example, we can check Axiom 7:

$$c \otimes (\mathbf{u} \oplus \mathbf{v}) = (\mathbf{uv})^c$$

while

$$(c \otimes \mathbf{u}) \oplus (c \otimes \mathbf{v}) = \mathbf{u}^c\mathbf{v}^c.$$

To make things work out, we find $\mathbf{0} = \mathbf{1}$, and $-\mathbf{u} = \mathbf{u}^{-1}$

What's going on here?

The following theorem is a direct consequence of the axioms.

Theorem

Let V be a vector space, $\mathbf{u}$ a vector in V and c a scalar.

1. $\mathbf{0}$ *is unique;*
2. $-\mathbf{u}$ *is the unique vector that satisfies* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$*;*
3. $0\mathbf{u} = \mathbf{0}$*; (note difference between* 0 *and* $\mathbf{0}$*)*
4. $c\mathbf{0} = \mathbf{0}$*;*
5. $(-1)\mathbf{u} = -\mathbf{u}$*.*

Exercises 4.1.25 - 29 of Lay outline the proofs of these results.

Subspaces

Some of the vector space examples we've seen "sit inside" others.
For example, we sketched the proof that $\mathbb{P}_2$ and $\mathbb{P}_4$ are both vector spaces. Any polynomial of degree at most two can also be viewed as a polynomial of degree at most 4:

$$a_0 + a_1 t + a_2 t^2 = a_0 + a_1 t + a_2 t^2 + 0t^3 + 0t^4.$$

If you have a subset H of a vector space V, some of the axioms are satisfied for free. For example, you don't need to check that scalar multiplication in H distributes through vector addition: you already know this is true in H because it's true in V.

Subspaces

This idea is formalised in the notion of a *subspace*.

Definition

A *subspace* of a vector space V is a subset H of V such that

1. The zero vector is in H: $\mathbf{0} \in H$;
2. whenever $\mathbf{u}, \mathbf{v}$ are in H, $\mathbf{u} + \mathbf{v}$ is in H.
   " H is closed under vector addition."
3. $c\mathbf{u}$ is in H whenever $\mathbf{u}$ is in H and c is in $\mathbb{R}$.
   "H is closed under scalar multiplication."

This is not a new idea: in MATH1013 the same definition is given for subspaces of $\mathbb{R}^n$.

Examples

Example 7

If V is any vector space, the subset $\{\mathbf{0}\}$ of V containing only the zero vector $\mathbf{0}$ is a subspace of V.

This is called the *zero subspace* or the *trivial subspace*.

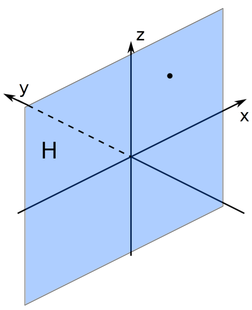
Example 8

Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Show that H is a subspace of $\mathbb{R}^3$.

- The zero vector of $\mathbb{R}^3$ is in H: set $a = 0$ and $b = 0$.
- H is closed under addition: adding two vectors in H always produces another vector whose second entry is 0 and therefore in H.
- H is closed under scalar multiplication: multiplying a vector in H by a scalar produces another vector in H.

Since all three properties hold, H is a subspace of $\mathbb{R}^3$.

If we identify vectors in $\mathbb{R}^3$ with points in 3D space as usual, then H is the plane through the origin given by the *homogeneous* equation $y = 0$.

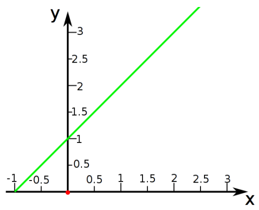


H is a plane, but H is NOT EQUAL to $\mathbb{R}^2$!
(The set $\mathbb{R}^2$ is not contained in $\mathbb{R}^3$.)

Example 9

Is $H = \left\{ \begin{bmatrix} s \\ s+1 \end{bmatrix} : s \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

We can identify H with the line whose equation is $y = x + 1$.



Clearly, the zero vector is not in H, so H is not a subspace of $\mathbb{R}^2$. (Observe that the equation $y = x + 1$ is *not* homogeneous).
As you saw in MATH1013, lines and planes through the origin are subspaces of $\mathbb{R}^n$ while lines and planes that do not pass through the origin are not subspaces.

Example 10

Let W be the set of symmetric 2×2 matrices:

$$W = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\} = \left\{ A \mid A^T = A \right\}.$$

Then W is a subspace of $M_{2\times2}$.

- The zero matrix satisfies the condition: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
- Let A and B be in W. Then $A^T = A$ and $B^T = B$, from which it follows that
$$(A + B)^T = A^T + B^T = A + B.$$
Therefore $A + B$ is symmetric and is in W.
- Similarly, $(cA)^T = cA^T = cA$, so cA is symmetric and is in W.

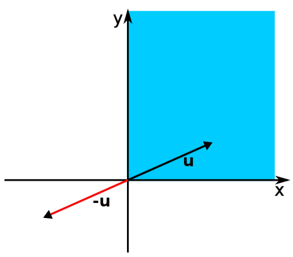
Example 11

Let V be the first quadrant in the xy-plane:

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}.$$

Is V a subspace of $\mathbb{R}^2$?

The answer is NO. Look at the picture below for example



Example 12

Let H be the set of all polynomials (with coefficients in $\mathbb{R}$) of degree at most two that have value 0 at $t = 1$

$$H = \{\mathbf{p} \in \mathbb{P}_2 \,:\, \mathbf{p}(1) = 0\}.$$

Is H a subspace of $\mathbb{P}_2$?

- The zero polynomial satisfies $\mathbf{0}(t) = 0$ for every t, so in particular $\mathbf{0}(1) = 0$.
- Let $\mathbf{p}$ and $\mathbf{q}$ be in H. Then $\mathbf{p}(1) = 0$ and $\mathbf{q}(1) = 0$
  Thus
  $$(\mathbf{p} + \mathbf{q})(1) = \mathbf{p}(1) + \mathbf{q}(1) = 0 + 0 = 0.$$
- If c is in $\mathbb{R}$ and $\mathbf{p}$ is in H we have
  $$(c\mathbf{p})(1) = c(\mathbf{p}(1)) = c0 = 0.$$

Yes, H is a subspace of $\mathbb{P}_2$!

Example 13

Let U be the set of all polynomials (with coefficients in $\mathbb{R}$) of degree at most two that have value 2 at $t = 1$

$$U = \{\mathbf{p} \in \mathbb{P}_2 \,:\, \mathbf{p}(1) = 2\}.$$

Is U a subspace of $\mathbb{P}_2$?

NO! In fact, the subset U doesn't satisfy any of the three subspace axioms.

Span: a recipe for building a subspace

Definition

Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in V, then the set of all vectors that can be written as linear combinations of the vectors is S is called Span(S):

$$\text{Span}(S) = \{c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p \,:\, c_1, \ldots, c_p \text{ are real numbers}\}$$

Theorem

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set of vectors in a vector space V. Then Span(S) is a subspace of V.

The subspace Span(S) is the "smallest" subspace of V that contains S, in the sense that if H is a subspace of V that contains all the vectors in S then Span$(S) \subset H$.

Example 14

Let $V = \{\langle a+3b, 3a-2b\rangle : a, b \in \mathbb{R}\}$. Is V a subspace of $\mathbb{R}^2$?

Write the vectors in V in column form:

$$\begin{aligned} \begin{bmatrix} a+3b \\ 3a-2b \end{bmatrix} &= \begin{bmatrix} a \\ 3a \end{bmatrix} + \begin{bmatrix} 3b \\ -2b \end{bmatrix} \\ &= a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 3 \\ -2 \end{bmatrix} \end{aligned}$$

So $V = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, and it is therefore a subspace of $\mathbb{R}^2$.
(In fact, it's all of $\mathbb{R}^2$, but that still counts as a subspace!)

Example 15

Let W be the set of all vectors in $\mathbb{R}^4$ of the form

$$\begin{bmatrix} 4a-2b \\ a+b+c \\ 0 \\ -2c-6a \end{bmatrix} \qquad (a, b, c \in \mathbb{R}) \qquad (W)$$

Show that W is a subspace of $\mathbb{R}^4$.

Since

$$\begin{bmatrix} 4a-2b \\ a+b+c \\ 0 \\ -2c-6a \end{bmatrix} = a\begin{bmatrix} 4 \\ 1 \\ 0 \\ -6 \end{bmatrix} + b\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix},$$

it follows that W is the subspace of $\mathbb{R}^4$ spanned by the three vectors

$$\begin{bmatrix} 4 \\ 1 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}.$$

Suggested exercises for review

Lay §4.1: 3, 9, 13, 33

Warm-up

Question

Do you understand the following sentence?

> *The set of 2×2 symmetric matrices is a subspace of the vector space of 2×2 matrices.*

Overview

Last time we defined an abstract vector space as a set of objects that satisfy 10 axioms. We saw that although $\mathbb{R}^n$ is a vector space, so is *the set of polynomials of a bounded degree* and *the set of all $n \times n$ matrices*. We also defined a *subspace* to be a subset of a vector space which is a vector space in its own right.

> *To check if a subset of a vector space is a subspace, you need to check that it contains the zero vector and is closed under addition and scalar multiplication.*

Recall from 1013 that a matrix has two special subspaces associated to it: the *null space* and the *column space*.

Question

How do the null space and column space generalise to abstract vector spaces?

(Lay, §4.2)

Matrices and systems of equations

Recall the relationship between a matrix and a system of linear equations:

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

The equation $A\mathbf{x} = \mathbf{b}$ corresponds to the system of equations

$$a_1x + a_2y + a_3z = b_1$$
$$a_4x + a_5y + a_6z = b_2.$$

We can find the solutions by row-reducing the augmented matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & b_1 \\ a_4 & a_5 & a_6 & b_2 \end{bmatrix}$$

to reduced echelon form.

The null space of a matrix

Let A be an $m \times n$ matrix.

Definition

The **null space** of A is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$:

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Example 1

Let $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}$.

Then the null space of A is the set of all scalar multiples of $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.

We can check easily that $A\mathbf{v} = \mathbf{0}$.
Furthermore, $A(t\mathbf{v}) = tA\mathbf{v} = t\mathbf{0} = \mathbf{0}$, so $t\mathbf{v} \in \text{Nul} A$.
To see that these are the *only* vectors in Nul A, solve the associated homogeneous system of equations.

The null space theorem

Theorem (Null Space is a Subspace)

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

This implies that the set of all solutions to a system of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

The null space theorem

Proof Since A has n columns, Nul A is a subset of $\mathbb{R}^n$. To show a subset is a subspace, recall that we must verify 3 axioms:

- $\mathbf{0} \in$ Nul A because $A\mathbf{0} = \mathbf{0}$.
- Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in Nul A. Then
$$A\mathbf{u} = \mathbf{0} \quad \text{and} \quad A\mathbf{v} = \mathbf{0}.$$
Therefore
$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$
This shows that $\mathbf{u}+\mathbf{v} \in$ Nul A.
- If c is any scalar, then
$$A(c\mathbf{u}) = c(A\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$
This shows that $c\mathbf{u} \in$ Nul A.

This proves that Nul A is a subspace of $\mathbb{R}^n$.

Example 2

Let $W = \left\{ \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} : \begin{array}{c} 3s - 4u = 5r + t \\ 3r + 2s - 5t = 4u \end{array} \right\}$ Show that W is a subspace.

Hint: Find a matrix A such that Nul A=W.

If we rearrange the equations given in the description of W we get

$$\begin{aligned} -5r + 3s - t - 4u &= 0 \\ 3r + 2s - 5t - 4u &= 0. \end{aligned}$$

So if A is the matrix $A = \begin{bmatrix} -5 & 3 & -1 & -4 \\ 3 & 2 & -5 & -4 \end{bmatrix}$, then W is the null space of A, and by the Null Space is a Subspace Theorem, W is a subspace of $\mathbb{R}^4$.

An explicit description of Nul A

The span of any set of vectors is a subspace. We can always find a spanning set for Nul A by solving the associated system of equations. (See Lay §1.5).

The column space of a matrix

Let A be an $m \times n$ matrix.

Definition

The **column space** of A is the set of all linear combinations of the columns of A.
If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\text{Col } A = \text{Span } \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}.$$

Theorem

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Why?

Example 3

Suppose

$$W = \left\{ \begin{bmatrix} 3a + 2b \\ 7a - 6b \\ -8b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Find a matrix A such that $W = \text{Col } A$.

$$W = \left\{ a \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ -6 \\ -8 \end{bmatrix} : a, b \in \mathbb{R} \right\} = \text{Span } \left\{ \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ -8 \end{bmatrix} \right\}$$

Put $A = \begin{bmatrix} 3 & 2 \\ 7 & -6 \\ 0 & -8 \end{bmatrix}$. Then $W = \text{Col } A$.

Another equivalent way to describe the column space is

$$\text{Col } A = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}.$$

Example 4

Let

$$\mathbf{u} = \begin{bmatrix} 6 \\ 7 \\ 1 \\ -4 \end{bmatrix}, \quad A = \begin{bmatrix} 5 & -5 & -9 \\ 8 & 8 & -6 \\ -5 & -9 & 3 \\ 3 & -2 & -7 \end{bmatrix}$$

Does $\mathbf{u}$ lie in the column space of A?

We just need to answer: *does $A\mathbf{x} = \mathbf{u}$ have a solution?*

Consider the following row reduction:

$$\left[\begin{array}{ccc|c} 5 & -5 & -9 & 6 \\ 8 & 8 & -6 & 7 \\ -5 & -9 & 3 & 1 \\ 3 & -2 & -7 & -4 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11/2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7/2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that the system $A\mathbf{x} = \mathbf{u}$ is consistent.

This means that the vector $\mathbf{u}$ can be written as a linear combination of the columns of A.

Thus $\mathbf{u}$ is contained in the Span of the columns of A, which is the column space of A. So the answer is YES!

Comparing Nul A and Col A

Example 5

Let $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$.

- The column space of A is a subspace of $\mathbb{R}^k$ where $k =$ ____.
- The null space of A is a subspace of $\mathbb{R}^k$ where $k =$ ____.
- Find a nonzero vector in Col A. (There are infinitely many.)
- Find a nonzero vector in Nul A.

For the final point, you may use the following row reduction:

$$\begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & -2 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{bmatrix}$$

Table: For any $m \times n$ matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	1.Col A is a subspace of $\mathbb{R}^m$.
2. Any $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	2. Any $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
3. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	3. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$.

Question

How does all this generalise to an abstract vector space?

An $m \times n$ matrix defines a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, and the null space and column space are subspaces of the domain and range, respectively.
We'd like to define the analogous notions for functions between arbitrary vector spaces.

Linear transformations

Definition

A *linear transformation* from a vector space V to a vector space W is a function $T : V \to W$ such that

L1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in V$;

L2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for $\mathbf{u} \in V, c \in \mathbb{R}$.

Matrix multiplication always defines a linear transfomation.

Example 6

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 4 \end{bmatrix}$. Then the mapping defined by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.
For example

$$T_A\left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$$

Example 7

Let $T : \mathbb{P}_2 \to \mathbb{P}_0$ be the map defined by

$$T(a_0 + a_1 t + a_2 t^2) = 2a_0.$$

Then T is a linear transformation.

$$\begin{aligned} T((a_0 + a_1 t + a_2 t^2) + (b_0 + b_1 t + b_2 t^2)) \\ &= T((a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2) \\ &= 2(a_0 + b_0) \\ &= 2a_0 + 2b_0 \\ &= T(a_0 + a_1 t + a_2 t^2) + T(b_0 + b_1 t + b_2 t^2). \end{aligned}$$

$$\begin{aligned} T(c(a_0 + a_1 t + a_2 t^2)) &= T(ca_0 + ca_1 t + ca_2 t^2) \\ &= 2ca_0 \\ &= cT(a_0 + a_1 t + a_2 t^2) \end{aligned}$$

Kernel of a linear transformation

Definition

The *kernel* of a linear transformation $T : V \to W$ is the set of all vectors $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$.
We write

$$\ker T = \{\mathbf{u} \in V : T(\mathbf{u}) = \mathbf{0}\}.$$

The kernel of a linear transformation T is analogous to the null space of a matrix, and $\ker T$ is a subspace of V.

If $\ker T = \{\mathbf{0}\}$, then T is *one to one*.

The range of a linear transformation

Definition

The *range* of a linear transformation $T : V \to W$ is the set of all vectors in W of the form $T(\mathbf{u})$ where $\mathbf{u}$ is in V.
We write

$$\text{Range } T = \{\mathbf{w} : \mathbf{w} = T(\mathbf{u}) \text{ for some } \mathbf{u} \in V\}.$$

The range of a linear transformation is analogous to the columns space of a matrix, and Range T is a subspace of W.

The linear transformation T is *onto* if its range is all of W.

Example 8

Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_0$ by

$$T(a_0 + a_1 t + a_2 t^2) = 2a_0.$$

Find the kernel and range of T.

The kernel consists of all the polynomials in $\mathbb{P}_2$ satisfying $2a_0 = 0$. This is the set

$$\{a_1 t + a_2 t^2\}.$$

The range of T is $\mathbb{P}_0$.

Example 9

The differential operator $D : \mathbb{P}_2 \to \mathbb{P}_1$ defined by $D(\mathbf{p}(x)) = \mathbf{p}'(x)$ is a linear transformation. Find its kernel and range.

First we see that

$$D(a + bx + cx^2) = b + 2cx.$$

So

$$\begin{aligned} \ker D &= \{a + bx + cx^2 : D(a + bx + cx^2) = 0\} \\ &= \{a + bx + cx^2 : b + 2cx = 0\} \end{aligned}$$

But $b + 2cx = 0$ if and only if $b = 2c = 0$, which implies $b = c = 0$. Therefore

$$\begin{aligned} \ker D &= \{a + bx + cx^2 : b = c = 0\} \\ &= \{a : a \in \mathbb{R}\} \end{aligned}$$

The range of D is all of $\mathbb{P}_1$ since every polynomial in $\mathbb{P}_1$ is the image under D (i.e the derivative) of *some* polynomial in $\mathbb{P}_2$.
To be more specific, if $a + bx$ is in $\mathbb{P}_1$, then

$$a + bx = D\left(ax + \frac{b}{2}x^2\right)$$

Example 10

Define $S : \mathbb{P}_2 \to \mathbb{R}^2$ by

$$S(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}.$$

That is, if $\mathbf{p}(x) = a + bx + cx^2$, we have

$$S(\mathbf{p}) = \begin{bmatrix} a \\ a+b+c \end{bmatrix}.$$

Show that S is a linear transformation and find its kernel and range.

Leaving the first part as an exercise to try on your own, we'll find the kernel and range of S.

- From what we have above, $\mathbf{p}$ is in the kernel of S if and only if

  $$S(\mathbf{p}) = \begin{bmatrix} a \\ a+b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  For this to occur we must have $a = 0$ and $c = -b$.
  So $\mathbf{p}$ is in the kernel of S if

  $$\mathbf{p}(x) = bx - bx^2 = b(x - x^2).$$

  This gives $\ker S = \text{Span } \{x - x^2\}$.

- The range of S.
  Since $S(\mathbf{p}) = \begin{bmatrix} a \\ a+b+c \end{bmatrix}$ and a, b and c are any real numbers, the range of S is all of $\mathbb{R}^2$.

Example 11

let $F : M_{2\times 2} \to M_{2\times 2}$ be the linear transformation defined by taking the transpose of the matrix:

$$F(A) = A^T.$$

We find the kernel and range of F.

We see that

$$\begin{aligned} \ker F &= \{A \text{ in } M_{2\times 2} : F(A) = 0\} \\ &= \{A \text{ in } M_{2\times 2} : A^T = 0\} \end{aligned}$$

But if $A^T = 0$, then $A = (A^T)^T = 0^T = 0$. It follows that $\ker F = 0$.
For any matrix A in $M_{2\times 2}$, we have $A = (A^T)^T = F(A^T)$. Since A^T is in $M_{2\times 2}$ we deduce that Range $F = M_{2\times 2}$.

Example 12

Let $S : \mathbb{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$S(\mathbf{p}(x)) = \int_0^1 \mathbf{p}(x)dx.$$

We find the kernel and range of S.

In detail, we have

$$\begin{aligned} S(a+bx) &= \int_0^1 (a+bx)dx \\ &= \left[ax + \frac{b}{2}x^2\right]_0^1 \\ &= a + \frac{b}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \ker S &= \{a+bx : S(a+bx) = 0\} \\ &= \left\{a+bx : a + \frac{b}{2} = 0\right\} \\ &= \left\{a+bx : a = -\frac{b}{2}\right\} \\ &= \left\{-\frac{b}{2} + bx\right\} \end{aligned}$$

Geometrically, $\ker S$ consists of all those linear polynomials whose graphs have the property that the area between the line and the x-axis is equally distributed above and below the axis on the interval $[0, 1]$.

The range of S is $\mathbb{R}$, since every number can be obtained as the image under S of some polynomial in $\mathbb{P}_1$.
For example, if a is an arbitrary real number, then

$$\int_0^1 a\ dx = [ax]_0^1 = a - 0 = a.$$

Overview

Last week we introduced the notion of an abstract vector space, and we saw that apparently different sets like polynomials, continuous functions, and symmetric matrices all satisfy the 10 axioms defining a vector space. We also discussed *subspaces*, subsets of a vector space which are vector spaces in their own right. To any **linear transformation** between vector spaces, one can associate two special subspaces:

- the kernel
- the range.

Today we'll talk about linearly independent vectors and bases for abstract vector spaces. The definitions are the same for abstract vector spaces as for Euclidean space, so you may find it helpful to review the material covered in 1013.

(Lay, §4.3, §4.4)

Linear independence

Definition (Linear Independence)

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be *linearly independent* if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \quad (1)$$

has *only* the trivial solution, $c_1 = c_2 = \cdots = c_p = 0$.

Definition

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be *linearly dependent* if it is not linearly independent, i.e., if there are some weights $c_1, c_2, \ldots, c_p$, **not all zero**, such that (1) holds.

Here's a recipe for proving a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent:

1. Write the equation
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$
2. Manipulate the equation to prove that all the $c_i = 0$. Done!
3. If you find a different solution, then you've instead proven that the set is linearly dependent.

!

If you start by assuming the c_i are all zero, you can't prove anything!

Example 1

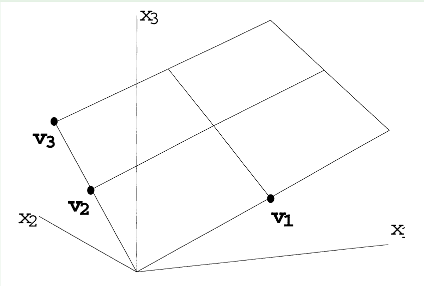
Show that the vectors $2x+3$, $4x^2$, and $1+x$ are linearly independent in $\mathbb{P}_2$.

1. Set a linear combination of the given vectors equal to $\mathbf{0}$:
$$a(2x+3)+b(4x^2)+c(1+x)=0.$$
2. Now manipulate the equation to see what coefficients are possible:
$$(3a+c)+(2a+c)x+4bx^2=0.$$
This implies
$$\begin{aligned}3a+c&=0\\2a+c&=0\\4b&=0\end{aligned}$$
But the only solution to this system is $a=b=c=0$, so the given vectors are linearly independent.

Span of a set

Example 2

Consider the plane H illustrated below:



Which of the following are valid descriptions of H?
(a) $H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2\}$ (b) $H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_3\}$
(c) $H = \text{Span}\ \{\mathbf{v}_2, \mathbf{v}_3\}$ (d) $H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2\ \mathbf{v}_3\}$

The spanning set theorem

Definition

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ equals H:
$$H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}.$$

Theorem (The spanning set theorem)

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$.

(a) *If the vector $\mathbf{v_k}$ in S is a linear combination of the remaining vectors of S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.*

(b) *If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.*

Example 3

Find a basis for $\mathbb{P}_2$ which is a subset of $S = \{1, x, 1+x, x+3, x^2\}$.

First, let's check if we have any hope: does S span $\mathbb{P}_2$?
The spanning set theorem says that if any vector in S is a linear combination of the other vectors in S, we can remove it without changing the span.

$$\text{Span}\ \{1, x, 1+x, x+3, x^2\} = \text{Span}\ \{1, x, x^2\}.$$

The set $\{1, x, x^2\}$ spans $\mathbb{P}_2$ and is linearly independent, so it's a basis.

Other correct answers are $\{1, 1+x, x^2\}$, $\{1, x+3, x^2\}$, $\{x+3, 1+x, x^2\}$, $\{x, x+3, x^2\}$, and $\{x, 1+x, x^2\}$.

Bases for Nul A and Col A

Given any subspace V, it's natural to ask for a basis of V.

When a subspace is defined as the null space or column space of a matrix, there is an algorithm for finding a basis.

Recall the following example from the last lecture:

Example 4

Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Row reducing the matrix gives

$$\begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r1 \to r1 - 5r2} \begin{bmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is equivalent to the system of equations

$$\begin{array}{ccccccccccc} x_1 & & & + & 6x_3 & - & 8x_4 & + & x_5 & = & 0 \\ & & x_2 & - & 2x_3 & + & x_4 & & & = & 0 \end{array}$$

The general solutions is $x_1 = -6x_3 + 8x_4 - x_5$, $x_2 = 2x_3 - x_4$. The free variables are x_3, x_4 and x_5.

We express the general solution in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_3 + 8x_4 - x_5 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_3 \underset{\substack{\uparrow \\ \mathbf{u}}}{\begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}} + x_5 \underset{\substack{\uparrow \\ \mathbf{w}}}{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

We get a vector for each free variable, and these form a spanning set for Nul A. In fact, this spanning set is linearly independent, so it's a basis.

A basis for Col A

Theorem

The pivot columns of a matrix A form a basis for Col A.

Although we won't prove this is true, we'll see why it should be plausible using this example.

Example 5

We find a basis for Col A, where

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 & -3 & 0 \\ 4 & 3 & 33 & -6 & 8 \\ 2 & -1 & 9 & -8 & -4 \\ -2 & 2 & -6 & 10 & 2 \end{bmatrix}$$

We row reduce A to get

$$A = \begin{bmatrix} 1 & 0 & 6 & -3 & 0 \\ 4 & 3 & 33 & -6 & 8 \\ 2 & -1 & 9 & -8 & -4 \\ -2 & 2 & -6 & 10 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 & -3 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_5 \end{bmatrix} \to \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix}$$

Note that

$$\mathbf{b}_3 = 6\mathbf{b}_1 + 3\mathbf{b}_2 \text{ and } \mathbf{b}_4 = -3\mathbf{b}_1 + 2\mathbf{b}_2$$

We can check that

$$\mathbf{a}_3 = 6\mathbf{a}_1 + 3\mathbf{a}_2 \text{ and } \mathbf{a}_4 = -3\mathbf{a}_1 + 2\mathbf{a}_2$$

Elementary row operations do not affect the linear dependence relationships among the columns of the matrix.

$$B = \begin{bmatrix} 1 & 0 & 6 & -3 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Looking at the columns of B, we can guess that $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_5}$ form a basis for Col B.
We check

1. $\mathbf{b}_2$ is not a multiple of $\mathbf{b}_1$.
2. $\mathbf{b}_5$ is not a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$.

Elementary row operations do not affect the linear dependence relationships among the columns of the matrix.

Since $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_5}\}$ is a basis for Col B,

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\} \text{ is a basis for Col } A.$$

Review

1. To find a basis for Nul A, use elementary row operations to transform $[A\ \mathbf{0}]$ to an equivalent reduced row echelon form $[B\ \mathbf{0}]$. Use the row reduced echelon form to find a parametric form of the general solution to $A\mathbf{x} = \mathbf{0}$. If Nul $A \neq \{\mathbf{0}\}$, the vectors found in this parametric form of the general solution are automatically linearly independent and form a basis for Nul A.
2. A basis for Col A is is formed from the pivot columns of A. The matrix B determines the pivot columns, but it is important to return to the matrix A.

The unique representation theorem

Theorem (The Unique Representation Theorem)

Suppose that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V. Then each $\mathbf{x} \in V$ has a unique expansion

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n \tag{2}$$

where $c_1, \ldots, c_n$ are in $\mathbb{R}^n$.

We say that the c_i are the *coordinates* of $\mathbf{x}$ relative to the basis $\mathcal{B}$, and we write $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Example 6

We found several bases for $\mathbb{P}_2$, including

$$\mathcal{B} = \{1, x, x^2\} \quad \text{and} \quad \mathcal{C} = \{1, x+3, x^2\}.$$

Find the coordinates for $5 + 2x + 3x^2$ with respect to $\mathcal{B}$ and $\mathcal{C}$.

We have

$$5 + 2x + 3x^2 = 5(1) + 2(x) + 3(x^2),$$

so $[5 + 2x + 3x^2]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$.

Similarly,

$$5 + 2x + 3x^2 = -1(1) + 2(x+3) + 3(x^2)$$

so $[5 + 2x + 3x^2]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

Why is the Unique Representation Theorem true?

Suppose that $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ is a basis for V, and that we can write

$$\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}$$
$$\mathbf{x} = d_1\mathbf{b_1} + \cdots + d_n\mathbf{b_n}.$$

We'd like to show that this implies $c_i = d_i$ for all i. Subtract the second line from the first to get

$$\mathbf{0} = (c_1 - d_1)\mathbf{b_1} + \cdots + (c_n - d_n)\mathbf{b_n}.$$

Since $\mathcal{B}$ is a basis, the $\mathbf{b_i}$ are linearly independent. This implies all the coefficients $c_i - d_i$ are equal to 0.
Thus, $c_i = d_i$ for all i.

Coordinates

Coordinates give instructions for writing a given vector as a linear combination of basis vectors.
In $\mathbb{R}^n$, we've been implicitly using the standard basis $\mathcal{E} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

.
However, we can express a vector in $\mathbb{R}^n$ in terms of any basis.

Example 7

Suppose $\mathcal{B} = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{E}}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{E}}\}$. Then $\mathbf{i} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{E}} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{E}}$, so $\mathbf{i} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}_{\mathcal{B}}$.

Overview

Last time we defined a *basis* of a vector space H:

Definition

The set $\{\mathbf{v_1}, \cdots, \mathbf{v_p}\}$ is a basis for H if

- $\{\mathbf{v_1}, \cdots, \mathbf{v_p}\}$ is linearly independent, and
- $\mathrm{Span}\{\mathbf{v_1}, \cdots, \mathbf{v_p}\} = H$

We recalled algorithms (§2.8, §4.3) to find a basis for the null space and the column space of a matrix, and we stated the Unique Representation Theorem:

> *Given a basis for H, every vector in H can be a written as a linear combination of basis vectors in a unique way.*

The coefficients of this expression are the *coordinates* of the vector with respect to the basis.

Question

Given bases $\mathcal{B}$ and $\mathcal{C}$ for H, how are $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$ related?

Coordinates

Theorem (The Unique Representation Theorem)

Suppose that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V. Then each $\mathbf{x} \in V$ has a unique expansion

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n \tag{1}$$

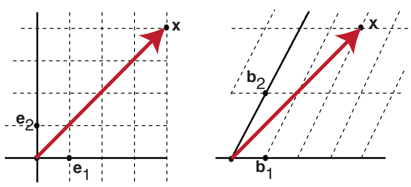
where $c_1, \ldots, c_n$ are in $\mathbb{R}$.

We say that the c_i are the *coordinates* of $\mathbf{x}$ relative to the basis $\mathcal{B}$, and we write $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Coordinates give instructions for writing a given vector as a linear combination of basis vectors.

Different bases determine different coordinates...

Suppose $\mathcal{B} = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{E}}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{E}}\}$, and as always, $\mathcal{E} = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{E}}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{E}}\}$.



Standard graph   *B*-graph paper

$$\text{If } [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{then} \quad \mathbf{x} = 2\mathbf{b}_1 + 2\mathbf{b}_2 = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{E}} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}_{\mathcal{E}}$$

$$\text{Similarly, } [\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \text{so} \quad \mathbf{x} = 4\mathbf{e}_1 + 4\mathbf{e}_2 = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{E}} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}_{\mathcal{E}}$$

...but some things stay the same

Even though we use different coordinates to describe the same point with respect to different bases, the structures we see in the vector space are independent of the chosen coordinates.

Definition

A one-to-one and onto linear transformation between vector spaces is an *isomorphism*. If there is an isomorphism $T : V_1 \to V_2$, we say that V_1 and V_2 are *isomorphic*.

Informally, we say that the vector space V is isomorphic to W if *every vector space calculation in V is accurately reproduced in W*, and vice versa.

For example, the property of a set of vectors being linearly independent doesn't depend on what coordinates they're written in.

Isomorphism

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $P : V \to \mathbb{R}^n$ defined by $P(\mathbf{x}) = [\mathbf{x}]_\mathcal{B}$ is an isomorphism.

What does this theorem mean?
V and $\mathbb{R}^n$ are both vector spaces, and we're defining a specific map that takes vectors in V to vectors in $\mathbb{R}^n$. This map

- ...is a linear transformation
- ...is one-to-one (i.e., if $P(\mathbf{u}) = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$)
- ...is onto (for every $\mathbf{v} \in \mathbb{R}^n$, there's some $\mathbf{u} \in V$ with $P(\mathbf{u}) = \mathbf{v}$)

Every vector space with an n-element basis is isomorphic to $\mathbb{R}^n$.

Very Important Consequences

If $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for a vector space V then

- A set of vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in V spans V if and only if the set of the coordinate vectors $\{[\mathbf{u}_1]_\mathcal{B}, \dots, [\mathbf{u}_p]_\mathcal{B}\}$ spans $\mathbb{R}^n$;
- A set of vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in V is linearly independent in V if and only if the set of the coordinate vectors $\{[\mathbf{u}_1]_\mathcal{B}, \dots, [\mathbf{u}_p]_\mathcal{B}\}$ is linearly independent in $\mathbb{R}^n$.
- An indexed set of vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in V is a basis for V if and only if the set of the coordinate vectors $\{[\mathbf{u}_1]_\mathcal{B}, \dots, [\mathbf{u}_p]_\mathcal{B}\}$ is a basis for $\mathbb{R}^n$.

Theorem

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors is linearly dependent.

Theorem

If a vector space V has a basis consisting of n vectors, then every basis of V must consist of exactly n vectors.

That is, every basis for V has the same number of elements. This number is called the *dimension* of V and we'll study it more tomorrow.

Changing Coordinates (Lay §4.7)

When a basis $\mathcal{B}$ is chosen for V, the associated coordinate mapping onto $\mathbb{R}^n$ defines a coordinate system for V. Each $\mathbf{x} \in V$ is identified uniquely by its coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

In some applications, a problem is initially described by using a basis $\mathcal{B}$, but by choosing a different basis $\mathcal{C}$, the problem can be greatly simplified and easily solved.

We want to study the relationship between $[\mathbf{x}]_{\mathcal{B}}, [\mathbf{x}]_{\mathcal{C}}$ in $\mathbb{R}^n$ and the vector $\mathbf{x}$ in V. We'll try to solve this problem in 2 different ways.

Changing from $\mathcal{B}$ to $\mathcal{C}$ coordinates: Approach #1

Example 1

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose that

$$\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2. \tag{2}$$

Further, suppose that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for some vector $\mathbf{x}$ in V. What is $[\mathbf{x}]_{\mathcal{C}}$?

Let's try to solve this from the definitions of the objects:

Since $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ we have

$$\mathbf{x} = 2\mathbf{b}_1 + 3\mathbf{b}_2. \tag{3}$$

The coordinate mapping determined by $\mathcal{C}$ is a linear transformation, so we can apply it to equation (3):

$$\begin{aligned}[\mathbf{x}]_{\mathcal{C}} &= [2\mathbf{b}_1 + 3\mathbf{b}_2]_{\mathcal{C}} \\ &= 2[\mathbf{b}_1]_{\mathcal{C}} + 3[\mathbf{b}_2]_{\mathcal{C}}\end{aligned}$$

We can write this vector equation as a matrix equation:

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}[\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}}\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}. \tag{4}$$

Here the vector $[\mathbf{b}_i]_{\mathcal{C}}$ becomes the i^{th} column of the matrix.
This formula gives us $[\mathbf{x}]_{\mathcal{C}}$ once we know the columns of the matrix. But from equation (2) we get

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix}-1\\4\end{bmatrix} \quad \text{and} \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix}5\\-3\end{bmatrix}$$

So the solution is

$$\begin{aligned}[\mathbf{x}]_{\mathcal{C}} &= \begin{bmatrix}-1 & 5\\4 & -3\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}13\\-1\end{bmatrix} \qquad \text{or} \\ [\mathbf{x}]_{\mathcal{C}} &= \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}\end{aligned}$$

where $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix}-1 & 5\\4 & -3\end{bmatrix}$ is called the *change of coordinate matrix from basis $\mathcal{B}$ to $\mathcal{C}$.*

Note that from equation (4), we have

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix}[\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}}\end{bmatrix}$$

The argument used to derive the formula (4) can be generalised to give the following result.

Theorem (2)

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases for a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}. \tag{5}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix}[\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}}\end{bmatrix}. \tag{6}$$

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in Theorem 12 is called the **change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.**
Multiplication by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.

Of course,

$$[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} [\mathbf{x}]_{\mathcal{C}} ,$$

so that

$$[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}},$$

whence $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are inverses of each other.

Summary of Approach #1

> *The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$.*

Why is this true, and what's a good way to remember this?

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ are bases for a vector space V. What is $[\mathbf{b}_1]_{\mathcal{B}}$?

$$[\mathbf{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We have

$$[\mathbf{b}_1]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{b}_1]_{\mathcal{B}},$$

so the first column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ needs to be the vector for $\mathbf{b}_1$ in $\mathcal{C}$ coordinates.

Example

Example 2

Find the change of coordinates matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ for the bases

$$\mathcal{B} = \{1, x, x^2\} \quad \text{and} \quad \mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$$

of $\mathbb{P}_2$.

Notice that it's "easy" to write a vector in $\mathcal{C}$ in $\mathcal{B}$ coordinates.

$$[1 + x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [x + x^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad [1 + x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Example 3 (continued)

Find the change of choordinates matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ for the bases

$$\mathcal{B} = \{1, x, x^2\} \quad \text{and} \quad \mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$$

of $\mathbb{P}_2$.

Since we just showed

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

we have

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}.$$

Suppose now that we have a polynomial $p(x) = 1 + 2x - 3x^2$ and we want to find its coordinates relative to the $\mathcal{C}$ basis.
We have

$$[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

and so

$$\begin{aligned} [p]_{\mathcal{C}} &= \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [p]_{\mathcal{B}} \\ &= \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}. \end{aligned}$$

Changing from $\mathcal{B}$ to $\mathcal{C}$ coordinates: Approach #2

As we just saw, it's relatively easy to find a change of basis matrix from a standard basis (e.g., $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{1, x, x^2, x^3\}$) to a non-standard basis.

We can use this fact to find a change of basis matrix between two non-standard bases, too. Suppose that $\mathcal{E}$ is a standard basis and $\mathcal{B}$ and $\mathcal{C}$ are non-standard bases for some vector space.
To change from $\mathcal{B}$ to $\mathcal{C}$ coordinates, first change from $\mathcal{B}$ to $\mathcal{E}$ coordinates and then change from $\mathcal{E}$ to $\mathcal{C}$ coordinates:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\mathbf{x} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P} \left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\mathbf{x} \right).$$

Since this is true for all $\mathbf{x}$, we can write the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ as a product of two matrices which are easy to find:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\,\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}.$$

Example 4

Consider the bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

We want to find the change of coordinate matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ using the method described above.

We have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 7 & 2 \\ -2 & -1 \end{bmatrix}, \quad \underset{\mathcal{E}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -5 \\ -1 & 4 \end{bmatrix}$$

Hence

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}^{-1} \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \frac{1}{3}\begin{bmatrix} 2 & -5 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$$

Examples: Approach #1

Example 5

Consider the bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We want to find the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$, and from $\mathcal{C}$ to $\mathcal{B}$.

Solution The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ involves the $\mathcal{C}$-coordinate vectors of $\mathbf{b}_1$ and $\mathbf{b}_2$. Suppose that

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

From the definition

$$\mathbf{b}_1 = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$\mathbf{b}_2 = y_1\mathbf{c}_1 + y_2\mathbf{c}_2 = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

To solve these systems simultaneously we augment the coefficient matrix with $\mathbf{b}_1$ and $\mathbf{b}_2$ and row reduce:

$$\begin{aligned} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} &= \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 4 & 1 & 8 & -5 \end{array}\right] \\ &\xrightarrow{rref} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{array}\right]. \end{aligned} \tag{7}$$

This gives

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad \text{and} \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix},$$

and

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

You may notice that the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ already appeared in (7). This is because the first column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ results from row reducing $\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{b}_1 \end{bmatrix}$ to $\begin{bmatrix} I & \vdots & [\mathbf{b}_1]_{\mathcal{C}} \end{bmatrix}$, and similarly for the second column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. Thus

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} I & \vdots & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{bmatrix}.$$

Example 6

Consider the bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

We want to find the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$, and from $\mathcal{C}$ to $\mathcal{B}$.

We use the following relationship:

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} I & \vdots & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{bmatrix}.$$

Here

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|cc} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{array}\right].$$

This gives

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}.$$

Further

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}.$$

Example 7

In $M_{2\times 2}$ let $\mathcal{B}$ be the basis

$$\left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and let $\mathcal{C}$ be the basis

$$\left\{ A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

We find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and verify that $[X]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[X]_{\mathcal{B}}$ for $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution To solve this problem directly we must find the coordinate vectors of $\mathcal{B}$ with respect to $\mathcal{C}$.

This would usually involve solving a system of 4 linear equations of the form $E_{11} = aA + bB + cC + dD$ where we need to find a, b, c and d.

We can avoid that in this case since we can find the required coefficients by inspection:
Clearly $E_{11} = A, E_{21} = -B + C, E_{12} = -A + B$ and $E_{22} = -C + D$.

Thus

$$[E_{11}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [E_{21}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, [E_{12}]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [E_{22}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

From this we have

$$\begin{aligned} \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} &= \begin{bmatrix} [E_{11}]_{\mathcal{C}} & [E_{21}]_{\mathcal{C}} & [E_{12}]_{\mathcal{C}} & [E_{22}]_{\mathcal{C}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

For $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,

$$X = 1E_{11} + 3E_{21} + 2E_{12} + 4E_{22}$$

and $[X]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}$.

We now want to verify that $[X]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [X]_{\mathcal{B}}$ for $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. From our calculations

$$\begin{aligned} [X]_{\mathcal{C}} &= \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [X]_{\mathcal{B}} \\ &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}. \end{aligned}$$

This is the coordinate vector of X with respect to the basis $\mathcal{C}$.

We check this as follows:

Since $[X]_{\mathcal{C}} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$ this means that X should be given by $-A - B - C + 4D$:

$$\begin{aligned} -A - B - C + 4D &= -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = X \end{aligned}$$

as it should be.

Overview

Given two bases $\mathcal{B}$ and $\mathcal{C}$ for the same vector space, we saw yesterday how to find the change of coordinates matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ nd $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$. Such a matrix is always square, since every basis for a vector space V has the same number of elements. Today we'll focus on this number —the *dimension* of V— and explore some of its properties.

From Lay, §4.5, 4.6

Dimension

Definition

If a vector space V is spanned by a finite set, then V is said to be **finite dimensional**.

The **dimension** of V, (written $\dim V$), is the number of vectors in a basis for V.

The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.

If V is not spanned by a finite set, then V is said to be **infinite dimensional**.

Example 1

1. The standard basis for $\mathbb{R}^n$ contains n vectors, so $\dim \mathbb{R}^n = n$.
2. The standard basis for $\mathbb{P}_3$, which is $\{1, t, t^2, t^3\}$, shows that $\dim \mathbb{P}_3 = 4$.
3. The vector space of continuous functions on the real line is infinite dimensional.

Dimension and the coordinate mapping

Recall the theorem we saw yesterday:

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $P : V \to \mathbb{R}^n$ defined by $P(\mathbf{x}) = [\mathbf{x}]_\mathcal{B}$ is an isomorphism.

(Recall that an isomorphism is a linear transformation that's both one-to-one and onto.)

This means that every vector space with an n-element basis is isomorphic to $\mathbb{R}^n$. We can now rephrase this theorem in new language:

Theorem

Any n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Dimensions of subspaces of $\mathbb{R}^3$

Example 2

- The **0 - dimensional subspace** contains only the zero vector $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$.
- If $\mathbf{u} \neq \mathbf{0}$, then Span $\{\mathbf{u}\}$ is a **1 - dimensional subspace**. These subspaces are **lines** through the origin.
- If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors in $\mathbb{R}^3$, then Span $\{\mathbf{u}, \mathbf{v}\}$ is a **2 - dimensional subspace**. These subspaces are **planes** through the origin.
- If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent vectors in $\mathbb{R}^3$, then Span $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a **3 - dimensional subspace**. This subspace is $\mathbb{R}^3$ itself.

Theorem

Let H be a subspace of a finite dimensional vector space V. Then any linearly independent set in H can be expanded (if necessary) to form a basis for H.

Also, H is finite dimensional and

$$\dim H \leq \dim V.$$

Example 3

Let $H = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$. Then H is a subspace of $\mathbb{R}^3$ and $\dim H < \dim \mathbb{R}^3$. Furthermore, we can expand the given spanning set for H $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ to

$$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

to form a basis for $\mathbb{R}^3$.

Question

Can you find another vector that you could have added to the spanning set for H to form a basis for $\mathbb{R}^3$?

When the dimension of a vector space or subspace is known, the search for a basis is simplified.

Theorem (The Basis Theorem)

Let V be a p-dimensional space, $p \geq 1$.

1. *Any linearly independent set of exactly p elements in V is a basis for V.*
2. *Any set of exactly p elements that spans V is a basis for V.*

Example 4

Schrödinger's equation is of fundamental importance in quantum mechanics. One of the first problems to solve is the one-dimensional equation for a simple quadratic potential, the so-called linear harmonic oscillator.
Analysing this leads to the equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

where $n = 0, 1, 2, \ldots$
There are polynomial solutions, the *Hermite polynomials*. The first few are

$$\begin{array}{ll} H_0(x) = 1 & H_3(x) = -12x + 8x^3 \\ H_1(x) = 2x & H_4(x) = 12 - 48x^3 + 16x^4 \\ H_2(x) = -2 + 4x^2 & H_5(x) = 120x - 160x^3 + 32x^5 \end{array}$$

We want to show that these polynomials form a basis for $\mathbb{P}_5$.

Writing the coordinate vectors relative to the standard basis for $\mathbb{P}_5$ we get

$$\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\4\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-12\\0\\8\\0\\0\end{bmatrix}, \begin{bmatrix}12\\0\\0\\-48\\16\\0\end{bmatrix}, \begin{bmatrix}0\\120\\0\\-160\\0\\32\end{bmatrix}.$$

This makes it clear that the vectors are linearly independent. Why?

Since $\dim \mathbb{P}_5 = 6$ and there are 6 polynomials that are linearly independent, the Basis Theorem shows that they form a basis for $\mathbb{P}_5$.

The dimensions of Nul A and Col A

Recall that last week we saw explicit algorithms for finding bases for the null space and the column space of a matrix A.

1. To find a basis for Nul A, use elementary row operations to transform $[A\ \mathbf{0}]$ to an equivalent reduced row echelon form $[B\ \mathbf{0}]$. Use the row reduced echelon form to find a parametric form of the general solution to $A\mathbf{x} = \mathbf{0}$. If Nul $A \neq \{\mathbf{0}\}$, the vectors found in this parametric form of the general solution are automatically linearly independent and form a basis for Nul A.
2. A basis for Col A is is formed from the pivot columns of A. The matrix B determines the pivot columns, but it is important to return to the matrix A.

Dimension of Nul A and Col A

The dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.
The dimension of Col A is the number of pivot columns in A.

Example 5

Given the matrix

$$A = \begin{bmatrix}1 & -6 & 9 & 10 & -2\\0 & 1 & 2 & -4 & 5\\0 & 0 & 0 & 5 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix},$$

what are the dimensions of the null space and column space?

There are three pivots and two free variables, so $\dim(\text{Nul } A) = 2$ and $\dim(\text{Col } A) = 3$.

Example 6

Given the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{bmatrix},$$

there are three pivots and no free variables, $\dim(\text{Nul } A) = 0$ and $\dim(\text{Col } A) = 3$.

The rank theorem

As before, let A be a matrix and let B be its reduced row echelon form

$$\dim \text{Col } A = \# \text{ of pivots of } A = \# \text{ of pivot columns of } B$$

Definition

The **rank** of a matrix A is the dimension of the column space of A.

$$\begin{aligned} \dim \text{Nul } A &= \# \text{ of free variables of } B \\ &= \# \text{ of non-pivot columns of } B. \end{aligned}$$

Compare the two red boxes. What does this tell about the relationship between the dimensions of the null space and column space of matrix?

Theorem

If A is an $m \times n$ matrix, then

$$\textit{Rank } A + \dim \textit{Nul } A = n.$$

Proof.

$$\left\{ \begin{array}{c} \text{number of} \\ \text{pivot columns} \end{array} \right\} + \left\{ \begin{array}{c} \text{number of} \\ \text{nonpivot columns} \end{array} \right\} = \left\{ \begin{array}{c} \text{number of} \\ \text{columns} \end{array} \right\}.$$

□

Examples

Example 7

If a 6×3 matrix A has rank 3, what can we say about $\dim \mathrm{Nul}\ A$, $\dim \mathrm{Col}\ A$ and Rank A?

- $\mathrm{Rank}\ A + \dim \mathrm{Nul}\ A = 3$.
- Since A only has three columns, and and all three are pivot columns, there are no free variables in the equation $A\mathbf{x} = \mathbf{0}$. Hence $\dim \mathrm{Nul}\ A = 0$.
- $\dim \mathrm{Col}\ A = \mathrm{Rank}\ A = 3$.

The row space of a matrix

The null space and the column space are the fundamental subspaces associated to a matrix, but there's one other natural subspace to consider:

Definition

The *row space* $\mathrm{Row}\ A$ of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

Example 8

For the matrix A given by

$$A = \begin{bmatrix} 1 & -6 & 9 & 10 & -2 \\ 3 & 1 & 2 & -4 & 5 \\ -2 & 0 & -1 & 5 & 1 \\ 4 & -3 & 1 & 0 & 6 \end{bmatrix},$$

we can write

$$\begin{aligned} \mathbf{r}_1 &= [1, -6, 9, 10, -2] \\ \mathbf{r}_2 &= [3, 1, 2, -4, 5] \\ \mathbf{r}_3 &= [-2, 0, -1, 5, 1] \\ \mathbf{r}_4 &= [4, -3, 1, 0, 6 \end{aligned}$$

The row space of A is the subspace of $\mathbb{R}^5$ spanned by $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$.

(Note that we're writing the vectors $\mathbf{r}_i$ as rows, rather than columns, for convenience.)

A basis for Row B

Theorem

Suppose a matrix B is obtained from a matrix A by row operations. Then $\mathrm{Row}\ A = \mathrm{Row}\ B$. If B is an echelon form of A, then the non-zero rows of B form a basis for $\mathrm{Row}\ B$.

Compare this to our procedure for finding a basis for Col A. Notice that it's simpler: after row reducing, we don't need to return to the original matrix to find our basis!

Proof.

If a matrix B is obtained from a matrix A by row operations, then the rows of B are linear combinations of those of A, so that $\mathrm{Row}\ B \subseteq \mathrm{Row}\ A$.
But row operations are reversible, which gives the reverse inclusion so that $\mathrm{Row}\ A = \mathrm{Row}\ B$.
In fact if B is an echelon form of A, then any non-zero row is linearly independent of the rows below it (because of the leading non-zero entry), and so the non-zero rows of B form a basis for $\mathrm{Row}\ B = \mathrm{Row}\ A$. □

The Rank Theorem –Updated!

Theorem

For any $m \times n$ matrix A, Col A and Row A have the same dimension. This common dimension, the rank of A, is equal to the number of pivot positions in A and satisfies the equation

$$\text{Rank } A + \dim \text{Nul } A = n.$$

This additional statement in this theorem follows from our process for finding bases for Row A and Col A:

Use row operations to replace A with its reduced row echelon form. Each pivot determines a vector (a column of A) in the basis for Col A and a vector (a row of B) in the basis for Row A.

Note also $\quad \text{Rank } A = \text{Rank } A^T.$

Example 9

Suppose a 4×7 matrix A has 4 pivot columns.

- Col $A \subseteq \mathbb{R}^4$ and $\dim$ Col $A = 4$. So Col $A = \mathbb{R}^4$.
- On the other hand, Row $A \subseteq \mathbb{R}^7$, so that even though $\dim$ Row $A = 4$, Row $A \neq \mathbb{R}^4$.

Example 10

If A is a 6×8 matrix, then the smallest possible dimension of Nul A is 2.

Example 11

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\{\mathbf{r}_1 = (1, 2, 0, 5), \mathbf{r}_2 = (0, 0, 1, -3)\}$ is a basis for $\mathrm{Row}\ A$.
(Note that these are rows of *rref*(A), not rows of A.)

Pivots are in columns 1 and 3 of *rref*(A), so that $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}\ A$. (Note these are columns of A.)

Example 12

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \xrightarrow{ref} B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of pivots in B is three, so $\dim \mathrm{Col}\ A = 3$ and a basis for Col A is given by

$$\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}$$

A basis for Row A is given by

$$\{(2, -3, 6, 2, 5), (0, 0, 3, -1, 1), (0, 0, 0, 1, 3)\}.$$

From B we can see that there are two free variables for the equation $A\mathbf{x} = \mathbf{0}$, so $\dim \mathrm{Nul}\ A = 2$. How would you find a basis for this subspace?

Applications to systems of equations

The rank theorem is a powerful tool for processing information about systems of linear equations.

Example 13

Suppose that the solutions of a homogeneous system of five linear equations in six unknowns are all multiples of one nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right hand side of the equations?

Solution The hardest thing to figure out is

What is the question asking?

A non-homogeneous system of equations $A\mathbf{x} = \mathbf{b}$ always has a solution if and only if the dimension of the column space of the matrix A is the same as the length of the columns.

In this case if we think of the system as $A\mathbf{x} = \mathbf{b}$, then A is a 5×6 matrix, and the columns have length 5: each column is a vector in $\mathbb{R}^5$.
The question is asking

Do the columns span $\mathbb{R}^5$?

or equivalently,

Is the rank of the column space equal to 5?

First note that $\dim \text{Nul } A = 1$. We use the equation:

$$\text{Rank } A + \dim \text{Nul } A = 6$$

to deduce that Rank $A = 5$.
Hence the dimension of the column space of A is 5, Col $A = \mathbb{R}^5$ and the system of non-homogeneous equations always has a solution.

Example 14

A homogeneous system of twelve linear equations in eight unknowns has two fixed solutions that are not multiples of each other, and all other solutions are linear combinations of these two solutions. Can the set of all solutions be described with fewer than twelve homogeneous linear equations? If so, how many?

Considering the corresponding matrix system $A\mathbf{x} = \mathbf{0}$, the key points are

- A is a 12×8 matrix.
- $\dim \text{Nul } A = 2$
- $\text{Rank } A + \dim \text{Nul } A = 8$
- What is the rank of A?
- How many equations are actually needed?

Example 15

Let $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$. The following are easily checked:

- Nul A is the z-axis.
- Row A is the xy-plane.
- Col A is the plane whose equation is $x + y = 0$.
- Nul A^T is the set of all multiples of $(1, 1, 0)$.
- Nul A and Row A are perpendicular to each other.
- Col A and Nul A^T are also perpendicular.

Theorem (Invertible Matrix Theorem ctd)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. *The columns of A form a basis of $\mathbb{R}^n$.*

n. *Col $A = \mathbb{R}^n$.*

o. dim *Col* $A = n$.

p. *Rank* $A = n$.

q. *Nul* $A = \{\mathbf{0}\}$.

r. dim *Nul* $A = 0$.

(The numbering continues the statement of the Invertible Matrix Theorem from Lay §2.3.)

Summary

1. Every basis for V has the same number of elements. This number is called the *dimension* of V.
2. If V is n-dimensional, V is isomorphic to $\mathbb{R}^n$.
3. A linearly independent list of vectors in V can be extended to a basis for V.
4. If the dimension of V is n, any linearly independent list of n vectors is a basis for V.
5. If the dimension of V is n, any spanning set of n vectors is a basis for V.

Applications to Markov chains

From Lay, §4.9
(This section is not examinable on the mid-semester exam.)

Theory and definitions

Markov chains are useful tools in certain kinds of probabilistic models. They make use of matrix algebra in a powerful way. The basic idea is the following: suppose that you are watching some collection of objects that are changing through time.

- Assume that the total number of objects is not changing, but rather their "states" (position, colour, disposition, etc) are changing.
- Further, assume that the proportion of state A objects changing to state B is constant and these changes occur at discrete stages, one after the next.

Then we are in a good position to model changes by a Markov chain.

As an example, consider the three storey aviary at a local zoo which houses 300 small birds. The aviary has three levels, and the birds spend their day flying around from one favourite perch to the next. Thus at any given time the birds seem to be randomly distributed throughout the three levels, except at feeding time when they all fly to the bottom level.

Our problem is to determine what the probability is of a given bird being at a given level of the aviary at a given time. Of course, the birds are always flying from one level to another, so the bird population on each level is constantly fluctuating. We shall use a Markov chain to model this situation.

Consider a 3×1 matrix

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

where p_1 is the percentage of total birds on the first level, p_2 is the percentage on the second level, and p_3 is the percentage on the third level. Note that $p_1 + p_2 + p_3 = 1 = 100\%$.
After 5 min we have a new matrix

$$\mathbf{p}' = \begin{bmatrix} p_1' \\ p_2' \\ p_3' \end{bmatrix}$$

giving a new distribution of the birds.

- We shall assume that the change from the $\mathbf{p}$ matrix to the $\mathbf{p}'$ matrix is given by a linear operator on $\mathbb{R}^3$.
- In other words there is a 3×3 matrix T, known as the **transition matrix** for the Markov chain, for which $T\mathbf{p} = \mathbf{p}'$.
- After another 5 minutes we have another distribution $\mathbf{p}'' = T\mathbf{p}'$ (using the same matrix T), and so forth.

The same matrix T is used since we are assuming that the probability of a bird moving to another level is independent of time.

In other words, the probability of a bird moving to a particular level depends only on the present state of the bird, and not on any past states —it's as if the birds had no memory of their past states.

This type of model is known as a finite Markov Chain.

A sequence of trials of an experiment is a finite Markov Chain if it has the following features:

- the outcome of each trials is one of a finite set of outcomes (such as {level 1, level 2, level 3} in the aviary example);
- the outcome of one trial depends only on the immediately preceding trial.

In order to give a more formal definition we need to introduce the appropriate terminology.

Definition

A vector $\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$ with nonnegative entries that add up to 1 is called a **probability vector**.

Definition

A **stochastic matrix** is a square matrix whose columns are probability vectors.

The transition matrix T described above that takes the system from one distribution to another is a stochastic matrix.

Definition

In general, a finite **Markov chain** is a sequence of probability vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ together with a stochastic matrix T, such that

$$\mathbf{x}_1 = T\mathbf{x}_0, \mathbf{x}_2 = T\mathbf{x}_1, \mathbf{x}_3 = T\mathbf{x}_2, \quad \cdots$$

We can rewrite the above conditions as a recurrence relation

$$\mathbf{x}_{k+1} = T\mathbf{x}_k, \quad \text{for } k = 0, 1, 2, \ldots$$

The vector $\mathbf{x}_k$ is often called a **state vector**.

More generally, a recurrence relation of the form

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \qquad \text{for} \qquad k = 0, 1, 2, \ldots$$

where A is an $n \times n$ matrix (not necessarily a stochastic matrix), and the $\mathbf{x}_k$s are vectors in $\mathbb{R}^n$ (not necessarily probability vector) is called a *first order difference equation*.

Examples

Example 1

We return to the aviary example. Assume that whenever a bird is on any level of the aviary, the probability of that bird being on the same level 5 min later is 1/2. If the bird is on the first level, the probability of moving to the second level in 5 min is 1/3 and of moving to the third level in 5 min is 1/6. For a bird on the second level, the probability of moving to either the first or third level is 1/4. Finally for a bird on the third level, the probability of moving to the second level is 1/3 and of moving to the first is 1/6.

We want to find the transition matrix for this example and use it to determine the distribution after certain periods of time.

From the information given, we derive the following matrix as the transition matrix:

$$\begin{array}{ccc} & & \text{From:} & & \\ & & \textit{lev } 1 \quad \textit{lev } 2 \quad \textit{lev } 3 & & \text{To:} \\ T & = & \begin{bmatrix} 1/2 & 1/4 & 1/6 \\ 1/3 & 1/2 & 1/3 \\ 1/6 & 1/4 & 1/2 \end{bmatrix} & \begin{array}{l} \textit{lev } 1 \\ \textit{lev } 2 \\ \textit{lev } 3 \end{array} \end{array}$$

Note that in each column, the sum of the probabilities is 1.

Using T we can now compute what happens to the bird distribution at 5-min intervals.

Suppose that immediately after breakfast all the birds are in the dining area on the first level. Where are they in 5 min? The probability matrix at time 0 is

$$\mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

According to the Markov chain model the bird distribution after 5 min is

$$T\mathbf{p} = \begin{bmatrix} 1/2 & 1/4 & 1/6 \\ 1/3 & 1/2 & 1/3 \\ 1/6 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix}$$

After another 5 min the bird distribution becomes

$$T \begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 13/36 \\ 7/18 \\ 1/4 \end{bmatrix}$$

Example 2

We investigate the weather in the Land of Oz. to illustrate the principles without too much heavy calculation.) The weather here is not ver good: there are never two fine days in a row.
If the weather on a particular day is known, we cannot predict exactly what the weather will be the next day, but we can predict the probabilities of various kinds of weather. We will say that there are only three kinds: fine, cloudy and rain.
Here is the behaviour:

- After a fine day, the weather is equally likely to be cloudy or rain.
- After a cloudy day, the probabilities are 1/4 fine, 1/4 cloudy and 1/2 rain.
- After rain, the probabilities are 1/4 fine, 1/2 cloudy and 1/4 rain.

We aim to find the transition matrix and use it to investigate some of the weather patterns in the Land of Oz.

The information gives a transition matrix:

$$\begin{array}{cc} & \text{From:} \\ & \begin{array}{ccc} \textit{fine} & \textit{cloudy} & \textit{rain} \end{array} \quad \text{To:} \\ T \;=\; & \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix} \begin{array}{c} \textit{fine} \\ \textit{cloudy} \\ \textit{rain} \end{array} \end{array}$$

Suppose on day 0 that the weather is rainy. That is

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then the probabilities for the weather the next day are

$$\mathbf{x}_1 = T\mathbf{x}_0 = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix},$$

and for the next day

$$\mathbf{x}_2 = T\mathbf{x}_1 = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 3/16 \\ 3/8 \\ 7/16 \end{bmatrix}$$

If we want to find the probabilities for the weather for a week after the initial rainy day, we can calculate like this

$$\mathbf{x}_7 = T\mathbf{x}_6 = T^2\mathbf{x}_5 = T^3\mathbf{x}_4 = \ldots = T^7\mathbf{x}_0.$$

Predicting the distant future

The most interesting aspect of Markov chains is the study of the chain's long term behaviour.

Example 3

Consider a system whose state is described by the Markov chain $\mathbf{x}_{k+1} = T\mathbf{x}_k$, for $k = 0, 1, 2, \ldots$, where T is the matrix

$$T = \begin{bmatrix} .7 & .2 & .2 \\ 0 & .2 & .4 \\ .3 & .6 & .4 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We want to investigate what happens to the system as time passes.

To do this we compute the state vector for several different times. We find

$$\mathbf{x}_1 = T\mathbf{x}_0 = \begin{bmatrix} .7 & .2 & .2 \\ 0 & .2 & .4 \\ .3 & .6 & .4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}$$

$$\mathbf{x}_2 = T\mathbf{x}_1 = \begin{bmatrix} .7 & .2 & .2 \\ 0 & .2 & .4 \\ .3 & .6 & .4 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.24 \\ 0.46 \end{bmatrix}$$

$$\mathbf{x}_3 = T\mathbf{x}_2 = \begin{bmatrix} .7 & .2 & .2 \\ 0 & .2 & .4 \\ .3 & .6 & .4 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.24 \\ 0.46 \end{bmatrix} = \begin{bmatrix} 0.350 \\ 0.232 \\ 0.416 \end{bmatrix}$$

Subsequent calculations give

$$\mathbf{x}_4 = \begin{bmatrix} 0.3750 \\ 0.2136 \\ 0.4114 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} 0.38750 \\ 0.20728 \\ 0.40522 \end{bmatrix},$$

$$\mathbf{x}_6 = \begin{bmatrix} 0.393750 \\ 0.203544 \\ 0.4027912 \end{bmatrix}, \quad \mathbf{x}_7 = \begin{bmatrix} 0.3968750 \\ 0.2017912 \\ 0.4013338 \end{bmatrix},$$

$$\mathbf{x}_8 = \begin{bmatrix} 0.39843750 \\ 0.20089176 \\ 0.4006704 \end{bmatrix}, \quad \mathbf{x}_9 = \begin{bmatrix} 0.399218750 \\ 0.200448848 \\ 0.400034602 \end{bmatrix},$$

$$\ldots, \mathbf{x}_{20} = \begin{bmatrix} 0.3999996185 \\ 0.2000002179 \\ 0.4000001634 \end{bmatrix}.$$

These vectors seem to be approaching

$$\mathbf{q} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix}.$$

Observe the following calculation:

$$T\mathbf{q} = \begin{bmatrix} .7 & .2 & .2 \\ 0 & .2 & .4 \\ .3 & .6 & .4 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix}.$$

This calculation is exact, with no rounding error. When the system is in state $\mathbf{q}$ there is no change in the system from one measurement to the next.
We might also note that T^{20} is given by

$$\begin{bmatrix} 0.4000005722 & 0.3999996185 & 0.3999996185 \\ 0.1999996730 & 0.2000002180 & 0.2000002179 \\ 0.3999997548 & 0.4000001635 & 0.4000001634 \end{bmatrix}.$$

Example 4

For the weather in the Land of Oz, where

$$T = \begin{bmatrix} 0 & 0.25 & 0.25 \\ 0.5 & 0.25 & 0.5 \\ 0.5 & 0.5 & 0.25 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have already calculated

$$\mathbf{x}_7 = \begin{bmatrix} 0.2000122070 \\ 0.4000244140 \\ 0.3999633789 \end{bmatrix}.$$

We want to look further ahead.

A further calculation gives

$$\mathbf{x}_{15} = \begin{bmatrix} 0.2000000002 \\ 0.4000000003 \\ 0.3999999994 \end{bmatrix}.$$

This suggests that

$$\mathbf{q} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}.$$

An easy calculation shows that $T\mathbf{q} = \mathbf{q}$.

Steady-state vectors

Definition

If T is a stochastic matrix, then a **steady state vector** for T is a probability vector $\mathbf{q}$ such that

$$T\mathbf{q} = \mathbf{q}.$$

A steady state vector $\mathbf{q}$ for T represents an *equilibrium* of the system modeled by the Markov Chain with transition matrix T. If at time 0 the system is in state $\mathbf{q}$ (that is if we have $\mathbf{x}_0 = \mathbf{q}$) then the system will remain in state $\mathbf{q}$ at all times (that is we will have $\mathbf{x}_n = \mathbf{q}$ for every $n \geq 0$).

It can be shown that every stochastic matrix has a steady state vector. In the examples in Section 2, the vector $\mathbf{q}$ is the steady state vector.
To find a suitable vector $\mathbf{q}$, we want to solve the equation $T\mathbf{x} = \mathbf{x}$.

$$\begin{aligned} T\mathbf{x} - \mathbf{x} &= \mathbf{0} \\ T\mathbf{x} - I\mathbf{x} &= \mathbf{0} \\ (T - I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

In the case $n = 2$, the problem is easily solved directly. Suppose first that all the entries of the transition matrix T are non-zero. Then T must be of the form

$$T = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix} \quad \text{for } 0 < p, q < 1.$$

Then

$$T - I = \begin{bmatrix} -p & q \\ p & -q \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} -p & q \\ 0 & 0 \end{bmatrix}.$$

So when solving $(T - I)\mathbf{x} = \mathbf{0}$, x_2 is free and $px_1 = qx_2$, so that

$$\mathbf{q} = \frac{1}{p+q}\begin{bmatrix} q \\ p \end{bmatrix}$$

is a steady state probability vector. Note that in this particular case the steady state vector is unique.

The case when one or more of the entries of T are zero is handled in a similar way. Note that if $p = q = 0$ then T is the identity matrix for which every probability vector is clearly a steady state vector.

A stochastic matrix does not necessarily have a *unique* steady state vector. In other words, a system modeled by a Markov Chain can have more than one equilibrium.

For example the probability vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

are all steady state vectors for the stochastic matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Indeed all the probability vectors

$$\begin{bmatrix} a \\ b \\ b \end{bmatrix} \quad \text{with } a, b \geq 0 \text{ and } a + 2b = 1$$

are steady state vectors for the above matrix T.

We would like to have some conditions on P that ensure that T has a unique steady state vector $\mathbf{q}$ and that the Markov Chain $\mathbf{x}_n$ associated to T converges to the steady state $\mathbf{q}$, independently of the initial state $\mathbf{x}_0$. For this kind of Markov chains, we can easily predict the long term behaviour.

It turns out that there is a large set of stochastic matrices for which long range predictions *are* possible. Before stating the main theorem we have to give a definition.

Definition

A stochastic matrix T is **regular** if some matrix power T^k contains only strictly positive entries.

In other words, if the transition matrix of a Markov chain is regular then, for some k, it is possible to go from any state to any state (including remaining in the current state) in exactly k steps.

For the transition matrix showing the probabilities for change in the weather in the Land of Oz, we have

$$T = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix}$$

However,

$$T^2 = \begin{bmatrix} 1/4 & 3/16 & 3/16 \\ 3/8 & 7/16 & 3/8 \\ 3/8 & 3/8 & 7/16 \end{bmatrix}$$

which shows that T is a regular stochastic matrix.

Here's an example of a stochastic matrix that is not regular:

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Not only does T have some zero entries , but also

$$T^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$T^3 = TT^2 = TI_2 = T$$

so that

$$T^k = T \quad \text{if } k \text{ is odd}, \qquad T^k = I_2 \quad \text{if } k \text{ is even.}$$

Thus any matrix power T^k has some entries equal to zero.

Theorem

If T is an $n \times n$ regular stochastic matrix, then T has a unique steady state vector $\mathbf{q}$. The entries of $\mathbf{q}$ are strictly positive

Moreover, if $\mathbf{x}_0$ is any initial probability vector and $\mathbf{x}_{k+1} = T\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$ then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{q}$ as $k \to \infty$.

Equivalently, the steady state vector $\mathbf{q}$ is the limit of $T^k\mathbf{x}_0$ when $k \to \infty$ for any probability vector $\mathbf{x}_0$.

Notice that if $T = [\mathbf{p}_1 \ldots \mathbf{p}_n]$, where $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are the columns of T, then taking $\mathbf{x}_0 = \mathbf{e}_i$, where $\mathbf{e}_i$ is the *ith* vector of the standard basis we have that

$$\mathbf{x}_1 = T\mathbf{x}_0 = T\mathbf{e}_i = \mathbf{p}_i$$

so $\mathbf{x}_1$ is the ith column of T.

Similarly $\mathbf{x}_k = T^k\mathbf{x}_0 = T^k\mathbf{e}_i$ is the ith column of T^k.

The previous theorem implies that $T^k\mathbf{e}_i \to \mathbf{q}$ for every $i = 1, \ldots, n$ when $k \to \infty$, that is every column of T^k approaches the limiting vector $\mathbf{q}$ when $k \to \infty$.

Examples

Example 5

Let $T = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$. We want to find the steady state vector associated with T.

We want to solve $(T - I)\mathbf{x} = \mathbf{0}$:

$$T - I = \begin{bmatrix} -0.2 & 0.5 \\ 0.2 & -0.5 \end{bmatrix} \to R = \begin{bmatrix} 1 & -5/2 \\ 0 & 0 \end{bmatrix}$$

The homogeneous system having the reduced row echelon matrix R as coefficient matrix is $x_1 - (5/2)x_2 = 0$. Taking x_2 as a free variable, the general solution is $x_1 = (5/2)t, \quad x_2 = t$.
For $\mathbf{x}$ to be a probability vector we also require $x_1 + x_2 = 1$.
Put $x_1 = (5/2)t, \ x_2 = t$, then $x_1 + x_2 = 1$ becomes $(5/2)t + t = 1$.
This gives $t = 2/7 = x_2$ and $x_1 = 5/7$, so $\mathbf{x} = \begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix}$.

An alternative Solution
If we consider $T = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$ as a matrix of the form

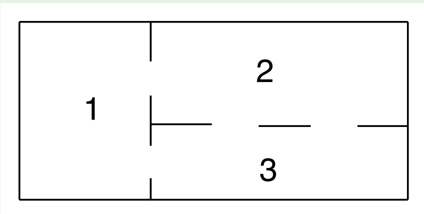
$$\begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}$$

we can identify $p = 0.2$ and $q = 0.5$. The solution is then given by

$$\mathbf{p} = \frac{1}{p+q}\begin{bmatrix} q \\ p \end{bmatrix} = \frac{1}{0.7}\begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix}.$$

Example 6

A psychologist places a rat in a cage with three compartments, as shown in the diagram.



The rat has been trained to select a door at random whenever a bell is rung and to move through it into the next compartment.

Example (continued)

From the diagram, if the rat is in space 1, there are equal probabilities that it will go to either space 2 or 3 (because there is just one opening to each of these spaces).

On the other hand, if the rat is in space 2, there is one door to space 1, and 2 to space 3, so the probability that it will go to space 1 is 1/3, and to space 3 is 2/3.

The situation is similar if the rat is in space 3. Wherever the rat is there is 0 probability that the rat will stay in that space.

The transition matrix is

$$P = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & 0 & 2/3 \\ 1/2 & 2/3 & 0 \end{bmatrix}.$$

It is easy to check that P^2 has entries which are strictly positive, so P is a regular stochastic matrix.
It is also easy to see that a rat can get from any room to any other room (including the one it starts from) through one or more moves.

To find the steady stat vector we need to solve $(P - I)\mathbf{x} = \mathbf{0}$, that is we need to find the null space of $P - I$.

$$\begin{aligned} P - I &= \begin{bmatrix} -1 & 1/3 & 1/3 \\ 1/2 & -1 & 2/3 \\ 1/2 & 2/3 & -1 \end{bmatrix} \\ &\xrightarrow{rref} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Hence if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $x_3 = t$ is free, $x_1 = \frac{2}{3}t, x_2 = t$. Since $\mathbf{x}$ must be a probability vector, we need $1 = x_1 + x_2 + x_3 = \frac{8}{3}t$. Thus, $t = \frac{3}{8}$ and

$$\mathbf{x} = \begin{bmatrix} 1/4 \\ 3/8 \\ 3/8 \end{bmatrix}.$$

In the long run, the rat spends $\frac{1}{4}$ of its time in space 1, and $\frac{3}{8}$ of its time in each of the other two spaces.

Eigenvectors and eigenvalues

From Lay, §5.1

Overview

Most of the material we've discussed so far falls loosely under two headings:

- geometry of $\mathbb{R}^n$, and
- generalisation of 1013 material to abstract vector spaces.

Today we'll begin our study of eigenvectors and eigenvalues. This is fundamentally different from material you've seen before, but we'll draw on the earlier material to help us understand this central concept in linear algebra. This is also one of the topics that you're most likely to see applied in other contexts.

Question

If you want to understand a linear transformation, what's the smallest amount of information that tells you something meaningful?

This is a very vague question, but studying eigenvalues and eigenvectors gives us one way to answer it.

From Lay, §5.1

Definition

An *eigenvector* of an $n \times n$ matrix A is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.
An *eigenvalue* of an $n \times n$ matrix A is a scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ has a non-zero solution; such a vector $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Example 1

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

Then any nonzero vector $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is an eigenvector for the eigenvalue 3:

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 3x \\ 0 \end{bmatrix}.$$

Similarly, any nonzero vector $\begin{bmatrix} 0 \\ y \end{bmatrix}$ is an eigenvector for the eigenvalue 2.

Sometimes it's not as obvious what the eigenvectors are.

Example 2

Let $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Then any nonzero vector $\begin{bmatrix} x \\ x \end{bmatrix}$ is an eigenvector for the eigenvalue 2:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 2x \\ 2x \end{bmatrix}.$$

Also, any nonzero vector $\begin{bmatrix} x \\ -x \end{bmatrix}$ is an eigenvector for the eigenvalue 0:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ -x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that an eigenvalue can be 0, but an eigenvector must be nonzero.

Eigenspaces

If λ is an eigenvalue of the $n \times n$ matrix A, we find corresponding eigenvectors by solving the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

The set of *all* solutions is just the null space of the matrix $A - \lambda I$.

Definition

Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the *zero* vector, is called the eigenspace of λ and is denoted by E_λ.

$$E_\lambda = \text{Nul } (A - \lambda I)$$

Example 3

As before, let $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. In the previous example, we verified that the given vectors were eigenvectors for the eigenvalues 2 and 0.
To find the eigenvectors for 2, solve for the null space of $B - 2I$:

$$\mathrm{Nul}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \mathrm{Nul}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} x \\ x \end{bmatrix}.$$

To find the eigenvectors for the eigenvalue 0, solve for the null space of $B - 0I = B$.

You can always check if you've correctly identified an eigenvector: simply multiply it by the matrix and make sure you get back a scalar multiple.

Eigenvalues of triangular matrix

Theorem

The eigenvalues of a triangular matrix A are the entries on the main diagonal.

Proof for the 3×3 *Upper Triangular Case*:
Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{33} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

Then

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{33} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}.$$

By definition, λ is an eigenvalue of A if and only if $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has non trivial solutions.
This occurs if and only if $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable.

Since

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

$(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable if and only if

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \text{or} \quad \lambda = a_{33}$$

An $n \times n$ matrix A has eigenvalue λ if and only if the equation

$$A\mathbf{x} = \lambda\mathbf{x}$$

has a nontrivial solution.

Equivalently, λ is an eigenvalue if $A - \lambda I$ is not invertible.
Thus, an $n \times n$ matrix A has eigenvalue $\lambda = 0$ if and only if the equation

$$A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$$

has a nontrivial solution.

This happens if and only if A is *not invertible*.

- The scalar 0 is an eigenvalue of A if and only if A is *not invertible*.

Theorem

Let A be an $n \times n$ matrix. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent.

The proof of this theorem is in Lay: Theorem 2, Section 5.1.

Example 4

Consider the matrix

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}.$$

We are given that A has an eigenvalue $\lambda = 3$ and we want to find a basis for the eigenspace E_3.

Solution We find the null space of $A - 3I$:

$$A - 3I = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$A - 3I \xrightarrow{rref} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So we get a single equation

$$x + 2y + 3z = 0 \quad \text{or} \quad x = -2y - 3z$$

and the general solution is

$$\mathbf{x} = \begin{bmatrix} -2y - 3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Hence $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for E_3.

Overview

The previous lecture introduced eigenvalues and eigenvectors. We'll review these definitions before considering the following question:

Question

Given a square matrix A, how can you find the eigenvalues of A?

We'll discuss an important tool for answering this question: the characteristic equation.

Lay, §5.2

Eigenvalues and eigenvectors

Definition

An *eigenvector* of an $n \times n$ matrix A is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is an *eigenvalue* for A.

Multiplying a vector by a matrix changes the vector. An eigenvector is a vector which is changed in the simplest way: by scaling.

Given any matrix, we can study the associated linear transformation. One way to understand this function is by identifying the set of vectors for which the transformation is just scalar multiplication.

Example

Example 1

Let $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$.

Then $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for the eigenvalue 2:

$$A\mathbf{u} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\mathbf{u}.$$

Also, $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector for the eigenvalue -1:

$$A\mathbf{v} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = -\mathbf{v}.$$

Finding Eigenvalues

Suppose we know that $\lambda \in \mathbb{R}$ is an eigenvalue for A. That is, for some $\mathbf{x} \neq \mathbf{0}$,

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Then we solve for an eigenvector $\mathbf{x}$ by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

But how do we find eigenvalues in the first place?

$$\begin{gathered} \mathbf{x} \text{ must be non zero} \\ \Downarrow \\ (A - \lambda I)\mathbf{x} = \mathbf{0} \text{ must have non trivial solutions} \\ \Downarrow \\ (A - \lambda I) \text{ is not invertible} \\ \Downarrow \\ \det(A - \lambda I) = 0. \end{gathered}$$

Solve $\det(A - \lambda I) = 0$ for λ to find the eigenvalues of the matrix A.

The eigenvalues of a square matrix A are the solutions of the characteristic equation.

the characteristic polynomial: $\det(A - \lambda I)$

the characteristic equation: $\det(A - \lambda I) = \mathbf{0}$

Examples

Example 2

Consider the matrix

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

We want to find the eigenvalues of A.

Since

$$A - \lambda I = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix},$$

The equation $\det(A - \lambda I) = \mathbf{0}$ becomes

$$\begin{aligned} (5-\lambda)(5-\lambda) - 9 &= 0 \\ \lambda^2 - 10\lambda + 16 &= 0 \\ (\lambda - 8)(\lambda - 2) &= 0 \\ \Rightarrow \lambda = 2, \lambda = 8. \end{aligned}$$

Example 3

Find the characteristic equation for the matrix

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

For a 3×3 matrix, recall that a determinant can be computed by cofactor expansion.

$$A - \lambda I = \begin{bmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{bmatrix}$$

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{bmatrix} \\
&= -\lambda \begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} - 3 \begin{vmatrix} 3 & 2 \\ 1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 3 & -\lambda \\ 1 & 2 \end{vmatrix} \\
&= -\lambda(\lambda^2 - 4) - 3(-3\lambda - 2) + (6 + \lambda) \\
&= -\lambda^3 + 4\lambda + 9\lambda + 6 + 6 + \lambda \\
&= -\lambda^3 + 14\lambda + 12
\end{aligned}$$

Hence the characteristic equation is

$$-\lambda^3 + 14\lambda + 12 = 0.$$

The eigenvalues of A are the solutions to the characteristic equation.

Example 4

Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -1 & 4 & 2 & 0 & 0 \\ 8 & 6 & -3 & 0 & 0 \\ 5 & -2 & 4 & -1 & 1 \end{bmatrix}$$

Find the characteristic equation for this matrix.

Observe that

$$\det(A-\lambda I) = \begin{bmatrix} 3-\lambda & 0 & 0 & 0 & 0 \\ 2 & 1-\lambda & 0 & 0 & 0 \\ -1 & 4 & 2-\lambda & 0 & 0 \\ 8 & 6 & -3 & -\lambda & 0 \\ 5 & -2 & 4 & -1 & 1-\lambda \end{bmatrix}$$

$$= (3-\lambda)(1-\lambda)(2-\lambda)(-\lambda)(1-\lambda)$$

$$= (-\lambda)(1-\lambda)^2(3-\lambda)(2-\lambda)$$

Thus A has eigenvalues $0, 1, 2$ and 3. The eigenvalue 1 is said to have *multiplicity* 2 because the factor $1-\lambda$ occurs twice in the characteristic polynomial.

In general the **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Similarity

The next theorem illustrates the use of the characteristic polynomial, and it provides a basis for several iterative methods that *approximate* eigenvalues.

Definition (Similar matrices)

If A and B are $n \times n$ matrices, then A is **similar** to B if there is an invertible matrix P such that

$$P^{-1}AP = B$$

or equivalently,

$$A = PBP^{-1}.$$

We say that A and B **are similar**. Changing A into $P^{-1}AP$ is called a **similarity transformation**.

Theorem

If the $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Proof.
If $B = P^{-1}AP$, then

$$\begin{aligned} B-\lambda I &= P^{-1}AP - \lambda P^{-1}P \\ &= P^{-1}(AP-\lambda P) \\ &= P^{-1}(A-\lambda I)P. \end{aligned}$$

Hence

$$\begin{aligned} \det(B-\lambda I) &= \det\left[P^{-1}(A-\lambda I)P\right] \\ &= \det(P^{-1})\det(A-\lambda I)\det P \\ &= \det(P^{-1})\det P\det(A-\lambda I) \\ &= \det(P^{-1}P)\det(A-\lambda I) \\ &= \det I\det(A-\lambda I) \end{aligned}$$

Application to dynamical systems

A dynamical system is a system described by a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$. Such an equation was used to model population movement in Lay 1.10 and it is the sort of equation used to model a Markov chain. Eigenvalues and eigenvectors provide a key to understanding the evolution of a dynamical system. Here's the idea that we'll see illustrated in the next example:

1. If you can, find a basis $\mathcal{B}$ of eigenvectors:
$$\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}.$$
2. Express the vector $\mathbf{x_0}$ describing the initial condition in $\mathcal{B}$ coordinates:
$$\mathbf{x_0} = c_1\mathbf{b_1} + c_2\mathbf{b_2}.$$
3. Since A multiplies each eigenvector by the corresponding eigenvalue, this makes it easy to see what happens after many iterations:
$$A^n\mathbf{x_0} = A^n(c_1\mathbf{b_1} + c_2\mathbf{b_2}) = c_1A^n\mathbf{b_1} + c_2A^n\mathbf{b_2} = c_1\lambda_1^n\mathbf{b_1} + c_2\lambda_2^n\mathbf{b_2}.$$

Examples

Example 5

In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 3% of the suburban population moves to the city. In 2000 there were 800,000 residents in the city and 500,000 residents in the suburbs. We want to investigate the result of this migration in the long term.

The migration matrix M is given by
$$M = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}.$$

- The first step is to find the eigenvalues of M.

The characteristic equation is given by
$$\begin{aligned} 0 &= \det \begin{bmatrix} .93-\lambda & .03 \\ .07 & .97-\lambda \end{bmatrix} \\ &= (.93-\lambda)(.97-\lambda) - (.03)(.07) \\ &= \lambda^2 - 1.9\lambda + .9021 - .0021 \\ &= \lambda^2 - 1.9\lambda + .9000 \\ &= (\lambda - 1)(\lambda - .9) \end{aligned}$$

So the eigenvalues are $\lambda = 1$ and $\lambda = 0.9$.
$$E_1 = \text{Nul} \begin{bmatrix} -.07 & .03 \\ .07 & -.03 \end{bmatrix} = \text{Nul} \begin{bmatrix} 7 & -3 \\ 0 & 0 \end{bmatrix}$$

This gives an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

$$E_{.9} = \text{Nul}\begin{bmatrix} .03 & .03 \\ .07 & .07 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and an eigenvector for this space is given by $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

- The next step is to write $\mathbf{x}_0$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

The initial vector $\mathbf{x}_0$ describes the initial population (in 2000), so writing in 100,000's we will put $\mathbf{x}_0 = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$.

There exist weights c_1 and c_2 such that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{1}$$

To find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ we do the following row reduction:

$$\begin{bmatrix} 3 & 1 & 8 \\ 7 & -1 & 5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1.3 \\ 0 & 1 & 4.1 \end{bmatrix}$$

So

$$\mathbf{x}_0 = 1.3\mathbf{v}_1 + 4.1\mathbf{v}_2. \tag{2}$$

- We can now look at the long term behaviour of the system. Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of M, with $M\mathbf{v}_1 = \mathbf{v}_1$ and $M\mathbf{v}_2 = .9\mathbf{v}_2$, we can compute each $\mathbf{x}_k$:

$$\begin{aligned} \mathbf{x}_1 = M\mathbf{x}_0 &= c_1M\mathbf{v}_1 + c_2M\mathbf{v}_2 \\ &= c_1\mathbf{v}_1 + c_2(0.9)\mathbf{v}_2 \\ \mathbf{x}_2 = M\mathbf{x}_1 &= c_1M\mathbf{v}_1 + c_2(0.9)M\mathbf{v}_2 \\ &= c_1\mathbf{v}_1 + c_2(0.9)^2\mathbf{v}_2 \end{aligned}$$

In general we have

$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(0.9)^k\mathbf{v}_2, \quad k = 0, 1, 2, \ldots,$$

that is

$$\mathbf{x}_k = 1.3\begin{bmatrix} 3 \\ 7 \end{bmatrix} + 4.1(0.9)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ k = 0, 1, 2, \ldots$$

As $k \to \infty$, $(0.9)^k \to 0$, and $\mathbf{x}_k \to 1.3\mathbf{v}_1$, which is $\begin{bmatrix} 3.9 \\ 9.1 \end{bmatrix}$. This indicates that in the long term 390,000 are expected to live in the city, while 910,000 are expected to live in the suburbs.

Example 6

Let $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$. We analyse the long-term behaviour of the dynamical system defined by $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $(k = 0, 1, 2, \ldots)$, with $\mathbf{x}_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$.

- As in the previous example we find the eigenvalues and eigenvectors of the matrix A.

$$\begin{aligned} 0 &= \det \begin{bmatrix} 0.8 - \lambda & 0.1 \\ 0.2 & 0.9 - \lambda \end{bmatrix} \\ &= (0.8 - \lambda)(0.9 - \lambda) - (0.1)(0.2) \\ &= \lambda^2 - 1.7\lambda + 0.7 \\ &= (\lambda - 1)(\lambda - 0.7) \end{aligned}$$

So the eigenvalues are $\lambda = 1$ and $\lambda = 0.7$. Eigenvalues corresponding to these eigenvalues are multiples of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

respectively. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is clearly a basis for $\mathbb{R}^2$.

- The next step is to write $\mathbf{x}_0$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

There exist weights c_1 and c_2 such that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{3}$$

To find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ we do the following row reduction:

$$\begin{bmatrix} 1 & 1 & 0.7 \\ 2 & -1 & 0.3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0.333 \\ 0 & 1 & 0.367 \end{bmatrix}$$

So

$$\mathbf{x}_0 = 0.333\mathbf{v}_1 + 0.367\mathbf{v}_2. \tag{4}$$

- We can now look at the long term behaviour of the system.

As in the previous example, since $\lambda_1 = 1$ and $\lambda_2 = 0.7$ we have

$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(0.7)^k\mathbf{v}_2, \quad k = 0, 1, 2, \ldots,$$

This gives

$$\mathbf{x}_k = 0.333 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.367(0.7)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ k = 0, 1, 2, \ldots$$

As $k \to \infty$, $(0.7)^k \to 0$, and $\mathbf{x}_k \to 0.333\mathbf{v}_1$, which is $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$. This is the steady state vector of the Markov chain described by A.

Some Numerical Notes

- Computer software such as Mathematica and Maple can use symbolic calculation to find the characteristic polynomial of a moderate sized matrix. There is no formula or finite algorithm to solve the characteristic equation of a general $n \times n$ matrix for $n \geq 5$.
- The best numerical methods for finding eigenvalues avoid the characteristic equation entirely. Several common algorithms for estimating eigenvalues are based on the Theorem on Similar matrices. Another technique, called *Jacobi's method* works when $A = A^T$ and computes a sequence of matrices of the form

  $$A_1 = A \text{ and } A_{k+1} = P_k^{-1}A_kP_k, \ k = 1, 2, \ldots.$$

  Each matrix in the sequence is similar to A and has the same eigenvalues as A. The non diagonal entries of A_{k+1} tend to 0 as k increases, and the diagonal entries tend to approach the eigenvalues of A.

Overview

In preparation for the exam, we'll look at the questions asked on the 2013 Mid-Semester Exam.

Sample Question: Lines & Planes

Let P be the plane in $\mathbb{R}^3$ defined by the equation $2x + y - z = 1$, and let L be the line through the point $(1, 1, 1)$ which is orthogonal to P.

1. Find an equation for P of the form $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ for some vector $\mathbf{n}$ and some vector $\mathbf{r_0}$.
2. Find an equation for L.
3. Let Q be the plane containing L and the point $(1, 1, 2)$. Find an equation for Q.

Solution: Lines & Planes

Let P be the plane in $\mathbb{R}^3$ defined by the equation $2x + y - z = 1$, and let L be the line through the point $(1, 1, 1)$ which is orthogonal to P.

1. Find an equation for P of the form $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ for some vector $\mathbf{n}$ and some vector $\mathbf{r_0}$.

To find the equation of a plane P, we need a **normal vector** to P and a **point** on P.

The plane $Ax + By + Cz + D = 0$ has normal vector $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$, so a normal vector to P is given by $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. To find a point on P, we can plug in $x = y = 0$ and see that $(0, 0, -1)$ satisfies the equation $2x + y - z = 1$. Thus the general formula $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ becomes

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} = 0.$$

Solution: Lines & Planes

Let P be the plane in $\mathbb{R}^3$ defined by the equation $2x + y - z = 1$, and let L be the line through the point $(1, 1, 1)$ which is orthogonal to P.

2. Find an equation for L.

A direction vector for L is any normal vector to P: i.e., any scalar multiple of $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. This yields the vector equation

$$\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix},$$

with the associated parametric equations

$$x = 1 + 2t \quad y = 1 + t \quad z = 1 - t.$$

Solution: Lines & Planes

Let P be the plane in $\mathbb{R}^3$ defined by the equation $2x + y - z = 1$, and let L be the line through the point $(1, 1, 1)$ which is orthogonal to P.

3. Let Q be the plane containing L and the point $(1, 1, 2)$. Find an equation for Q.

To find a normal vector to the new plane, take the cross product of two vectors parallel to Q. For example, you could choose a direction vector for L and the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ between the two given points on Q:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j}.$$

Any equation for the plane is acceptable, including the following:

$$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = 0,$$

Sample Question: Bases & Coordinates

The set $\mathcal{B} = \{t + 1, 1 + t^2, 3 - t^2\}$ is a basis for $\mathbb{P}_2$.

1. If $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, express p in the form $p(t) = a + bt + ct^2$.
2. Find the coordinate vector of the polynomial $q(t) = 2 - 2t$ with respect to $\mathcal{B}$ coordinates.

Solution: Bases & Coordinates

The set $\mathcal{B} = \{t+1, 1+t^2, 3-t^2\}$ is a basis for $\mathbb{P}_2$.

1. If $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, express p in the form $p(t) = a + bt + ct^2$.

Since the $\mathcal{B}$ coordinates of p are $1, 1$, and -1, we have

$$p(t) = 1(t+1) + 1(1+t^2) - 1(3-t^2) = -1 + t + 2t^2.$$

Solution: Bases & Coordinates

The set $\mathcal{B} = \{t+1, 1+t^2, 3-t^2\}$ is a basis for $\mathbb{P}_2$.

2. Find the coordinate vector of the polynomial $q(t) = 2 - 2t$ with respect to $\mathcal{B}$ coordinates.

We need a, b, and c such that

$$a(t+1) + b(1+t^2) + c(3-t^2) = 2 - 2t.$$

Collecting like powers of t gives us a system of equations:

$$\begin{aligned} a + b + 3c &= 2 \\ a &= -2 \\ b - c &= 0. \end{aligned}$$

The unique solution to this is $a = -2$, $b = c = 1$.
To protect against algebra mistakes, check that

$$-2(t+1) + 1(1+t^2) + 1(3-t^2) = 2 - 2t.$$

Sample Question: Vector Spaces

Decide whether each of the following sets is a vector space. If it is a vector space, state its dimension. If it is not a vector space, explain why.

1. A is the set of 2×2 matrices whose entries are integers.
2. B is the set of vectors in $\mathbb{R}^3$ which are orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.
3. C is the set of polynomials whose derivative is 0:

$$C = \{p(x) \in \mathbb{P} \mid \frac{d}{dx}p(x) = 0\}.$$

Solution: Vector Spaces

Decide whether each of the following sets is a vector space. If it is a vector space, state its dimension. If it is not a vector space, explain why.

1. A is the set of 2×2 matrices whose entries are integers.

This is a subset of the vector space of 2×2 matrices with real entries, so we can check if the three subspace axioms hold:

1. Is 0 in the set?
2. Is the set closed under addition?
3. Is the set closed under scalar multiplication?

No, this is not a vector space. This set is not closed under multiplication by a non-integer scalar. For example,

$$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \text{ is not in } A.$$

Solution: Vector Spaces

Decide whether each of the following sets is a vector space. If it is a vector space, state its dimension. If it is not a vector space, explain why.

2. B is the set of vectors in $\mathbb{R}^3$ which are orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

As before, we could check the 3 subspace axioms, but it's quicker to observe that B is the null space of the matrix $[1 \;\; 0 \;\; 2]$, and the null space of a matrix is always a subspace.
We can find a basis for the null space explicitly and check that it has 2 vectors. Alternatively, observe that the matrix $[1 \;\; 0 \;\; 2]$ has rank 1, so its null space is two-dimensional by the Rank Theorem.

Checking the 3 subspace axioms

1. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0$, so $\mathbf{0} \in B$.
2. Suppose $\mathbf{v}, \mathbf{u} \in B$. Then $\mathbf{v} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \mathbf{u} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0.$

$$(\mathbf{u}+\mathbf{v}) \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \mathbf{u} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \mathbf{v} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0 + 0 = 0.$$

   Since $\mathbf{u}+\mathbf{v}$ is in B, B is closed under addition.
3. Suppose $\mathbf{v} \in B$.

$$(c\mathbf{v}) \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = c\left(\mathbf{v} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) = c0 = 0.$$

   Since $c\mathbf{v}$ is in B, B is closed under scalar multiplication.

Solution: Vector Spaces

Decide whether each of the following sets is a vector space. If it is a vector space, state its dimension. If it is not a vector space, explain why.

3. the set of polynomials whose derivative is 0:

$$C = \left\{ p(x) \in \mathbb{P} \;\middle|\; \frac{d}{dx} p(x) = 0 \right\}.$$

We can solve this problem by recognising that the polynomials whose derivatives are 0 are exactly the constant polynomials, so $C = \mathbb{R}^1$. It follows that C is a one-dimensional vector space.

It is also acceptable to show that C is a subspace of the vector space $\mathbb{P}$ by verifying each of the subspace axioms.

Sample Question: Linear transformations

A linear transformation $T : M_{2\times 2} \to M_{2\times 2}$ is defined by:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

(a) Calculate $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.

(b) Which, if any, of the following matrices are in $\ker(T)$?

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

(c) Which, if any, of the following matrices are in $\text{range}(T)$?

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Find the kernel of T and explain why T is not one to one.

(e) Explain why T does not map $M_{2\times 2}$ onto $M_{2\times 2}$.

Sample Question: Subspaces associated to a matrix

Consider the matrix A:

$$\begin{bmatrix} 2 & -4 & 0 & 2 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 1 & 4 \end{bmatrix}.$$

(i) Find a basis for $\text{Nul}\, A$.

(ii) Find a basis for $\text{Col}\, A$.

(iii) Consider the linear transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$. Give a geometric description of the range of T_A as a subspace of $\mathbb{R}^3$. What is its dimension? Does it pass through the origin?

We begin by row-reducing A:

$$\begin{bmatrix} 2 & -4 & 0 & 2 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 1 & 4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Find a basis for Nul A.

The general solution to $R \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = 0$ is $y + 3z = 0$, $w - 2x + z = 0$, so

$$\text{Nul}\, A = \left\{ \begin{bmatrix} 2x - z \\ x \\ -3z \\ z \end{bmatrix} \right\} = \left\{ x \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

and so $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A.

We begin by row-reducing A:

$$\begin{bmatrix} 2 & -4 & 0 & 2 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 1 & 4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(ii) Find a basis for Col A.

A basis for Col A is obtained by taking every column of A that corresponds to a pivot column in the row reduced form of A. Thus the first and third columns

$$\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

form a basis for Col A.

(iii) Consider the linear transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$. Give a geometric description of the range of T_A as a subspace of $\mathbb{R}^3$. What is its dimension? Does it pass through the origin?

The range of T_A is exactly the column space of A. We just saw that it has a basis with two elements, so it is two dimensional. It is a plane in $\mathbb{R}^3$, and passed through the origin, because every vector subspace contains $\mathbf{0}$.

Revision: Definitions

- What is a vector space? Give some examples.
- What is a subspace? How do you check if a subset of a vector space is a subspace?
- What is a linear transformation? Give some examples.
- What does it mean for a set of vectors to be linearly independent? How do you check this?
- What are the coordinates of a vector with respect to a basis?

Revision: Geometry of $\mathbb{R}^3$

- What information do you need to determine a line? A plane?
- How can you check if two lines are orthogonal? Parallel?
- How do you find the distance between a point and a line? A point and a plane?
- How can you find the angle between two vectors?
- What are the scalar and vector projections of one vector onto another? Can you describe these in words?

Revision: Bases

- What is a basis for a vector space?
- If the dimension of V is n, then V and $\mathbb{R}^n$ are *isomorphic*. What does this mean and how do we know it's true?
- In an n-dimensional vector space,
  - any n linearly independent vectors form a basis.
  - any n vectors which span V form a basis.
  - any set of vectors which spans V contains a basis for V.
  - any set of linearly independent vectors can be extended to a basis for V.
- How do you find a basis for the null space of a matrix? The column space? The row space? The kernel of the associated linear transformation? (Which pair of these are the same?)

Overview

Before the break, we began to study eigenvectors and eigenvalues, introducing the characteristic equation as a tool for finding the eigenvalues of a matrix:

$$\det(A - \lambda I) = 0.$$

The roots of the characteristic equation are the eigenvalues of λ.
We also discussed the notion of similarity: the matrices A and B are *similar* if $A = PBP^{-1}$ for some invertible matrix P.

Question

When is a matrix A similar to a diagonal matrix?

From Lay, §5.3

Quick review

Definition

An *eigenvector* of an $n \times n$ matrix A is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is an *eigenvalue* for A.

To find the eigenvalues of a matrix, find the solutions of the characteristic equation:

$$\det(A - \lambda I) = \mathbf{0}.$$

The λ-eigenspace is the set of all eigenvectors for the eigenvalue λ, together with the zero vector. The λ-eigenspace E_λ is Nul $(A - \lambda I)$.

The advantages of a diagonal matrix

Given a diagonal matrix, it's easy to answer the following questions:

1. What are the eigenvalues of D? The dimensions of each eigenspace?
2. What is the determinant of D?
3. Is D invertible?
4. What is the characteristic polynomial of D?
5. What is D^k for $k = 1, 2, 3, \dots$?

For example, let $D = \begin{bmatrix} 10^{50} & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & -2.7 \end{bmatrix}$.

Can you answer each of the questions above?

The diagonalisation theorem

The goal in this section is to develop a useful factorisation $A = PDP^{-1}$, for an $n \times n$ matrix A. This factorisation has several advantages:

- it makes transparent the geometric action of the associated linear transformation, and
- it permits easy calculation of A^k for large values of k:

Example 1

Let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Then the transformation T_D scales the three standard basis vectors by $2, -4$, and -1, respectively.

$$D^7 = \begin{bmatrix} 2^7 & 0 & 0 \\ 0 & (-4)^7 & 0 \\ 0 & 0 & (-1)^7 \end{bmatrix}.$$

Example 2

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. We will use similarity to find a formula for A^k. Suppose we're given $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$.

We have

$$\begin{aligned} A &= PDP^{-1} \\ A^2 &= PDP^{-1}PDP^{-1} \\ &= PD^2P^{-1} \\ A^3 &= PD^2P^{-1}PDP^{-1} \\ &= PD^3P^{-1} \\ \vdots & \quad \vdots \quad \vdots \\ A^k &= PD^kP^{-1} \end{aligned}$$

So

$$A^k = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 4^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 2/5 & 3/5 \\ 1/5 & -1/5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5}4^k + \frac{3}{5}(-1)^k & \frac{3}{5}4^k - \frac{3}{5}(-1)^k \\ \frac{2}{5}4^k - \frac{2}{5}(-1)^k & \frac{3}{5}4^k + \frac{2}{5}(-1)^k \end{bmatrix}$$

Diagonalisable Matrices

Definition

An $n \times n$ (square) matrix is **diagonalisable** if there is a diagonal matrix D such that A is similar to D.

That is, A is diagonalisable if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$ (or equivalently $A = PDP^{-1}$).

Question

How can we tell when A is diagonalisable?

The answer lies in examining the eigenvalues and eigenvectors of A.

Recall that in Example 2 we had

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \text{ and } A = PDP^{-1}.$$

Note that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$A \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = -1 \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

We see that each column of the matrix P is an eigenvector of A...

This means that we can view P as a change of basis matrix from eigenvector coordinates to standard coordinates!

In general, if $AP = PD$, then

$$A \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

If $\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$ is invertible, then A is the same as

$$\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}^{-1}.$$

Theorem (The Diagonalisation Theorem)

Let A be an $n \times n$ matrix. Then A is diagonalisable if and only if A has n linearly independent eigenvectors.

$P^{-1}AP$ is a diagonal matrix D if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors of A in the same order.

Example 1

Find a matrix P that diagonalises the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

- The characteristic polynomial is given by

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{bmatrix}. \\ &= (-1-\lambda)(-\lambda)(-1-\lambda) + \lambda \\ &= -\lambda^2(\lambda + 2). \end{aligned}$$

The eigenvalues of A are $\lambda = 0$ (of multiplicity 2) and $\lambda = -2$ (of multiplicity 1).

- The eigenspace E_0 has a basis consisting of the vectors

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and the eigenspace E_{-2} has a basis consisting of the vector

$$\mathbf{p}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

It is easy to check that these vectors are linearly independent.

- So if we take

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

then P is invertible.

It is easy to check that $AP = PD$ where $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$$AP = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}.$$

Example 2

Can you find a matrix P that diagonalises the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}?$$

- The characteristic polynomial is given by

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{bmatrix} \\ &= (-\lambda)\left[-\lambda(4-\lambda)+5\right] - 1(-2) \\ &= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 \\ &= -(\lambda-1)^2(\lambda-2) \end{aligned}$$

This means that A has eigenvalues $\lambda = 1$ (of multiplicity 2) and $\lambda = 2$ (of multiplicity 1).

- The corresponding eigenspaces are

$$E_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, E_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}.$$

Note that although $\lambda = 1$ has multiplicity 2, the corresponding eigenspace has dimension 1. This means that we can only find 2 linearly independent eigenvectors, and by the Diagonalisation Theorem A is not diagonalisable.

Example 3

Consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Why is A diagonalisable?

Since A is upper triangular, it's easy to see that it has three distinct eigenvalues: $\lambda_1 = 2, \lambda_2 = 5$ and $\lambda_3 = 1$. Eigenvectors corresponding to distinct eigenvalues are linearly independent, so A has three linearly independent eigenvectors and is therefore diagonalisable.

Theorem

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalisable.

Example 4

Is the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

diagonalisable?

The eigenvalues are $\lambda = 4$ with multiplicity 2, and $\lambda = 2$ with multiplicity 2.

The eigenspace E_4 is found as follows:

$$\begin{aligned} E_4 &= \text{Nul} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \\ &= \text{Span} \left\{ \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \end{aligned}$$

and has dimension 2.

The eigenspace E_2 is given by

$$E_2 = \text{Nul}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \text{Span}\left\{ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and has dimension 2.

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is linearly independent.

This implies that $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$ is invertible and $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$.

1. *For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to its multiplicity.*
2. *The matrix A is diagonalisable if and only if the sum of the dimensions of the distinct eigenspaces equals n.*
3. *If A is diagonalisable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.*
4. *If $P^{-1}AP = D$ for a diagonal matrix D, then P is the change of basis matrix from eigenvector coordinates to standard coordinates.*

Overview

Last week introduced the important Diagonalisation Theorem:

> *An $n \times n$ matrix A is diagonalisable if and only if there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.*

This week we'll continue our study of eigenvectors and eigenvalues, but instead of focusing just on the matrix, we'll consider the associated linear transformation.

From Lay, §5.4

Question

If we always treat a matrix as defining a linear transformation, what role does diagonalisation play?

(The version of the lecture notes posted online has more examples than will be covered in class.)

Introduction

We know that a matrix determines a linear transformation, but the converse is also true:
if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T can be obtained as a matrix transformation

$$(*) \qquad T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

for a unique matrix A.
To construct this matrix, define

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \cdots\ T(\mathbf{e}_n)],$$

the $m \times n$ matrix whose columns are the images via T of the vectors of the standard basis for $\mathbb{R}^n$ (notice that $T(\mathbf{e}_i)$ is a vector in $\mathbb{R}^m$ for every $i = 1, \ldots, n$).

The matrix A is called the *standard matrix* of T.

Example 1

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by the formula

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ 3x + y \\ x - y \end{bmatrix}.$$

Find the standard matrix of T.

The standard matrix of T is the matrix $[[T(\mathbf{e}_1)]_{\mathcal{E}}\ [T(\mathbf{e}_2)]_{\mathcal{E}}]$.
Since

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \qquad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix},$$

the standard matrix of T is the 3×2 matrix

$$\begin{bmatrix} 1 & -1 \\ 3 & 1 \\ 1 & -1 \end{bmatrix}.$$

Example 2

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. What does the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ do to each of the standard basis vectors?

- The image of $\mathbf{e_1}$ is the vector $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = T(\mathbf{e_1})$. Thus, we see that T multiplies any vector parallel to the x-axis by the scalar 2.
- The image of $\mathbf{e_2}$ is the vector $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = T(\mathbf{e_2})$. Thus, we see that T multiplies any vector parallel to the y-axis by the scalar -1.
- The image of $\mathbf{e_3}$ is the vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = T(\mathbf{e_3})$. Thus, we see that T sends a vector parallel to the z-axis to a vector with equal x and z coordinates.

When we introduced the notion of coordinates, we noted that choosing different bases for our vector space gave us different coordinates. For example, suppose

$$\mathcal{E} = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\} \text{ and } \mathcal{B} = \{\mathbf{e_1}, \mathbf{e_2}, -\mathbf{e_1} + \mathbf{e_3}\}.$$

Then

$$\mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{B}}.$$

When we say that $T\mathbf{x} = A\mathbf{x}$, we are implicitly assuming that everything is written in terms of standard $\mathcal{E}$ coordinates.

Instead, it's more precise to write

$$[T(\mathbf{x})]_{\mathcal{E}} = A[\mathbf{x}]_{\mathcal{E}} \quad \text{with} \quad A = [[T(\mathbf{e}_1)]_{\mathcal{E}} \ [T(\mathbf{e}_2)]_{\mathcal{E}} \ \cdots \ [T(\mathbf{e}_n)]_{\mathcal{E}}]$$

Every linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be described as multiplication by its standard matrix: the standard matrix of T describes the action of T in terms of the coordinate systems on $\mathbb{R}^n$ and $\mathbb{R}^m$ given by the standard bases of these spaces.

If we start with a vector expressed in $\mathcal{E}$ coordinates, then it's convenient to represent the linear transformation T by $[T(\mathbf{x})]_{\mathcal{E}} = A[\mathbf{x}]_{\mathcal{E}}$.

However, for any sets of coordinates on the domain and codomain, we can find a matrix that represents the linear transformation in those coordinates:

$$[T(\mathbf{x})]_{\mathcal{C}} = A[\mathbf{x}]_{\mathcal{B}}$$

(Note that the domain and codomain can be described using different coordinates! This is obvious when A is an $m \times n$ matrix for $m \neq n$, but it's also true for linear transformations from $\mathbb{R}^n$ to itself.)

Example 3

For $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we saw that $[T(\mathbf{x})]_{\mathcal{E}} = A[\mathbf{x}]_{\mathcal{E}}$ acted as follows:

- T multiplies any vector parallel to the x-axis by the scalar 2.
- T multiplies any vector parallel to the y-axis by the scalar -1.
- T sends a vector parallel to the z-axis to a vector with equal x and z coordinates.

Describe the matrix B such that $[T(\mathbf{x})]_{\mathcal{B}} = A[\mathbf{x}]_{\mathcal{B}}$, where $\mathcal{B} = \{\mathbf{e_1}, \mathbf{e_2}, -\mathbf{e_1} + \mathbf{e_3}\}$.

Just as the i^{th} column of A is $[T(\mathbf{e_i})]_{\mathcal{E}}$, the i^{th} column of B will be $[T(\mathbf{b_i})]_{\mathcal{B}}$.

Since $\mathbf{e_1} = \mathbf{b_1}$, $T(\mathbf{b_1}) = 2\mathbf{b_1}$. Similarly, $T(\mathbf{b_2}) = -\mathbf{b_2}$.

Thus we see that $B = \begin{bmatrix} 2 & 0 & * \\ 0 & -1 & * \\ 0 & 0 & * \end{bmatrix}$.

The third column is the interesting one. Again, recall $\mathcal{B} = \{\mathbf{e_1}, \mathbf{e_2}, -\mathbf{e_1} + \mathbf{e_3}\}$, and

- T multiplies any vector parallel to the x-axis by the scalar 2.
- T multiplies any vector parallel to the y-axis by the scalar -1.
- T sends a vector parallel to the z-axis to a vector with equal x and z coordinates.

The 3^{rd} column of B will be $[T(\mathbf{b_3})]_{\mathcal{B}}$.

$$T(\mathbf{b_3}) = T(-\mathbf{e_1}+\mathbf{e_3}) = -T(\mathbf{e_1})+T(\mathbf{e_3}) = -2\mathbf{e_1}+(\mathbf{e_1}+\mathbf{e_3}) = -\mathbf{e_1}+\mathbf{e_3} = \mathbf{b_3}.$$

Thus we see that $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Notice that in B coordinates, the matrix representing T is diagonal!

Every linear transformation $T : V \to W$ between finite dimensional vector spaces can be represented by a matrix, but the matrix representation of a linear transformation depends on the choice of bases for V and W (thus it is not unique).

This allows us to reduce many linear algebra problems concerning abstract vector spaces to linear algebra problems concerning the familiar vector spaces $\mathbb{R}^n$. This is important even for linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ since certain choices of bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ can make important properties of T more evident: to solve certain problems easily, it is important to choose the *right* coordinates.

Matrices and linear transformations

Let $T : V \to W$ be a linear transformation that maps from V to W, and suppose that we've fixed a basis $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ for V and a basis $\mathcal{C} = \{\mathbf{c_1}, \ldots, \mathbf{c_m}\}$ for W.

For any vector $\mathbf{x} \in V$, the coordinate vector $[\mathbf{x}]_\mathcal{B}$ is in $\mathbb{R}^n$ and the coordinate vector of its image $[T(\mathbf{x})]_\mathcal{C}$ is in $\mathbb{R}^m$.

We want to associate a matrix M with T with the property that $M[\mathbf{x}]_\mathcal{B} = [T(\mathbf{x})]_\mathcal{C}$.

It can be helpful to organise this information with a diagram

$$\begin{array}{ccc} V \ni \mathbf{x} & \xrightarrow{\quad T \quad} & T(\mathbf{x}) \in W \\ \downarrow & & \downarrow \\ \mathbb{R}^n \ni [\mathbf{x}]_\mathcal{B} & \xrightarrow[\text{multiplication by } M]{} & [T(\mathbf{x})]_\mathcal{C} \in \mathbb{R}^m \end{array}$$

where the vertical arrows represent the coordinate mappings.

Here's an example to illustrate how we might find such a matrix M:
Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for two vector spaces V and W, respectively.
Let $T : V \to W$ be the linear transformation defined by

$$T(\mathbf{b}_1) = 2\mathbf{c}_1 - 3\mathbf{c}_2, \; T(\mathbf{b}_2) = -4\mathbf{c}_1 + 5\mathbf{c}_2\,.$$

Why does this define the entire linear transformation? For an arbitrary

vector $\mathbf{v} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2$ in V, we define its image under T as

$$T(\mathbf{v}) = x_1 T(\mathbf{b}_1) + x_2 T(\mathbf{b}_2).$$

For example, if $\mathbf{x}$ is the vector in V given by $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$, so that $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, we have

$$\begin{aligned} T(\mathbf{x}) &= T(3\mathbf{b}_1 + 2\mathbf{b}_2) \\ &= 3T(\mathbf{b}_1) + 2T(\mathbf{b}_2) \\ &= 3(2\mathbf{c}_1 - 3\mathbf{c}_2) + 2(-4\mathbf{c}_1 + 5\mathbf{c}_2) \\ &= -2\mathbf{c}_1 + \mathbf{c}_2. \end{aligned}$$

Equivalently, we have

$$\begin{aligned}[T(\mathbf{x})]_{\mathcal{C}} &= [3T(\mathbf{b}_1) + 2T(\mathbf{b}_2)]_{\mathcal{C}} \\ &= 3[T(\mathbf{b}_1)]_{\mathcal{C}} + 2[T(\mathbf{b}_2)]_{\mathcal{C}} \\ &= \begin{bmatrix}[T(\mathbf{b}_1)]_{\mathcal{C}} & [T(\mathbf{b}_2)]_{\mathcal{C}}\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} \\ &= \begin{bmatrix}[T(\mathbf{b}_1)]_{\mathcal{C}} & [T(\mathbf{b}_2)]_{\mathcal{C}}\end{bmatrix}[\mathbf{x}]_{\mathcal{B}}\end{aligned}$$

In this case, since $T(\mathbf{b}_1) = 2\mathbf{c}_1 - 3\mathbf{c}_2$ and $T(\mathbf{b}_2) = -4\mathbf{c}_1 + 5\mathbf{c}_2$ we have

$$[T(\mathbf{b}_1)]_{\mathcal{C}} = \begin{bmatrix}2\\-3\end{bmatrix} \quad \text{and} \quad [T(\mathbf{b}_2)]_{\mathcal{C}} = \begin{bmatrix}-4\\5\end{bmatrix}$$

and so

$$\begin{aligned}[T(\mathbf{x})]_{\mathcal{C}} &= \begin{bmatrix}2 & -4\\-3 & 5\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} \\ &= \begin{bmatrix}-2\\1\end{bmatrix}.\end{aligned}$$

In the last page, we are not so much interested in the actual calculation but in the equation

$$[T(\mathbf{x})]_{\mathcal{C}} = \begin{bmatrix}[T(\mathbf{b}_1)]_{\mathcal{C}} & [T(\mathbf{b}_2)]_{\mathcal{C}}\end{bmatrix}[\mathbf{x}]_{\mathcal{B}}$$

This gives us the matrix M:

$$M = \begin{bmatrix}[T(\mathbf{b}_1)]_{\mathcal{C}} & [T(\mathbf{b}_2)]_{\mathcal{C}}\end{bmatrix}$$

whose columns consist of the coordinate vectors of $T(\mathbf{b}_1)$ and $T(\mathbf{b}_2)$ with respect to the basis $\mathcal{C}$ in W.

In general, when T is a linear transformation that maps from V to W where $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_m\}$ is a basis for W the matrix associated to T with respect to these bases is

$$M = \begin{bmatrix}[T(\mathbf{b}_1)]_{\mathcal{C}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{C}}\end{bmatrix}.$$

We write $\underset{\mathcal{C}\leftarrow\mathcal{B}}{T}$ for M, so that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{T}$ has the property

$$\begin{aligned}[T(\mathbf{x})]_{\mathcal{C}} &= \begin{bmatrix}[T(\mathbf{b}_1)]_{\mathcal{C}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{C}}\end{bmatrix}[\mathbf{x}]_{\mathcal{B}} \\ &= \underset{\mathcal{C}\leftarrow\mathcal{B}}{T}[\mathbf{x}]_{\mathcal{B}}.\end{aligned}$$

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{T}$ describes how the linear transformation T operates in terms of the coordinate systems on V and W associated to the basis $\mathcal{B}$ and $\mathcal{C}$ respectively.

NB. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{T}$ is *the matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$*. It depends on the choice of *both* the bases $\mathcal{B}, \mathcal{C}$. The order of $\mathcal{B}$, $\mathcal{C}$ is important.

In the case that $T : V \to V$ and $\mathcal{B} = \mathcal{C}$, $\underset{\mathcal{B}\leftarrow\mathcal{B}}{T}$ is written $[T]_{\mathcal{B}}$ and is the *matrix for T relative to $\mathcal{B}$*, or more shortly the $\mathcal{B}$-matrix of T.

So by taking bases in each space, and writing vectors with respect to these bases, T can be studied by studying the matrix associated to T with respect to these bases.

Algorithm for finding the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{T}$

To find the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{T}$ where $T : V \to W$ relative to

a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of V

a basis $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ of W

- Find $T(\mathbf{b}_1), T(\mathbf{b}_2), \ldots, T(\mathbf{b}_n)$.
- Find the coordinate vector $[T(\mathbf{b}_1)]_{\mathcal{C}}$ of $T(\mathbf{b}_1)$ with respect to the basis $\mathcal{C}$. This is a column vector in $\mathbb{R}^m$.
- Do this for each $T(\mathbf{b}_i)$.
- Make a matrix from these column vectors. This matrix is $\underset{\mathcal{C}\leftarrow\mathcal{B}}{T}$.

N.B. The coordinate vectors $[T(\mathbf{b}_1)]_{\mathcal{C}}, [T(\mathbf{b}_2)]_{\mathcal{C}}, \ldots, [T(\mathbf{b}_n)]_{\mathcal{C}}$ have to be written as columns (not rows!).

Examples

Example 4

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for vector spaces V and W respectively. $T : V \to W$ is the linear transformation with the property that

$$\begin{aligned} T(\mathbf{b}_1) &= 3\mathbf{d}_1 - 5\mathbf{d}_2, \\ T(\mathbf{b}_2) &= -\mathbf{d}_1 + 6\mathbf{d}_2, \\ T(\mathbf{b}_3) &= 4\mathbf{d}_2 \end{aligned}$$

We find the matrix $\underset{\mathcal{D}\leftarrow\mathcal{B}}{T}$ of T relative to $\mathcal{B}$ and $\mathcal{D}$.

We have

$$[T(\mathbf{b}_1)]_{\mathcal{D}} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, [T(\mathbf{b}_2)]_{\mathcal{D}} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

and

$$[T(\mathbf{b}_3)]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

This gives

$$\underset{\mathcal{D}\leftarrow\mathcal{B}}{T} = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{D}} & [T(\mathbf{b}_3)]_{\mathcal{D}} & [T(\mathbf{b}_3)]_{\mathcal{D}} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}.$$

Example 5

Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by

$$T(p(t)) = \begin{bmatrix} p(0) + p(1) \\ p(-1) \end{bmatrix}.$$

(a) Show that T is a linear transformation.

(b) Find the matrix $\underset{\mathcal{E}\leftarrow\mathcal{B}}{T}$ of T relative to the standard bases $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{P}_2$ and $\mathbb{R}^2$.

(a) This is an exercise for you.

combinations of the vectors in $\mathcal{E}$).

$$T(1) = \begin{bmatrix} 1+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\mathbf{e}_1 + \mathbf{e}_2$$
$$T(t) = \begin{bmatrix} 0+1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mathbf{e}_1 - \mathbf{e}_2$$
$$T(t^2) = \begin{bmatrix} 0+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{e}_1 + \mathbf{e}_2.$$

• STEP 2 We find the coordinate vectors of $T(1)$, $T(t)$, $T(t^2)$ in the basis $\mathcal{E}$:

$$[T(1)]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad [T(t)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad [T(t^2)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

• STEP 3 We form the matrix whose columns are the coordinate vectors in step 2

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{T} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Example 6

Let $V = \mathrm{Span}\{\sin t, \cos t\}$, and $D : V \to V$ the linear transformation $D : f \mapsto f'$. Let $\mathbf{b}_1 = \sin t, \mathbf{b}_2 = \cos t$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, a basis for V. We find the matrix of T with respect to the basis $\mathcal{B}$.

- STEP 1 We have

$$D(\mathbf{b}_1) = \cos t = 0\mathbf{b}_1 + 1\mathbf{b}_2,$$

$$D(\mathbf{b}_2) = -\sin t = -1\mathbf{b}_1 + 0\mathbf{b}_2.$$

- STEP 2 From this we have

$$[D(\mathbf{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [D(\mathbf{b}_2)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

- STEP 3 So that

$$[D]_{\mathcal{B}} = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let $f(t) = 4\sin t - 6\cos t$. We can use the matrix we have just found to get the derivative of $f(t)$. Now $[f(t)]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$. Then

$$\begin{aligned} [D(f(t))]_{\mathcal{B}} &= [D]_{\mathcal{B}}[f(t)]_{\mathcal{B}} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 4 \end{bmatrix}. \end{aligned}$$

This, of course gives

$$f'(t) = 6\sin t + 4\cos t$$

which is what we would expect.

Example 7

Let $M_{2\times2}$ be the vector space of 2×2 matrixes and let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Let $T : M_{2\times2} \to \mathbb{P}_2$ be the linear transformation given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + b + c + (a - c)x + (a + d)x^2.$$

We find the matrix of T with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for $M_{2\times2}$ and the standard basis $\mathcal{C} = \{1, x, x^2\}$ for $\mathbb{P}_2$.

- STEP 1 We find the effect of T on each of the basis elements:

$$T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right) = 1+x+x^2,$$
$$T\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right) = 1,$$
$$T\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right) = 1-x,$$
$$T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = x^2.$$

- STEP 2 The corresponding coordinate vectors are

$$\left[T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix}1\\1\\1\end{bmatrix},$$
$$\left[T\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix}1\\0\\0\end{bmatrix},$$
$$\left[T\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix}1\\-1\\0\end{bmatrix},$$
$$\left[T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

- STEP 3 Hence the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$\begin{bmatrix}1&1&1&0\\1&0&-1&0\\1&0&0&1\end{bmatrix}.$$

Example 8

We consider the linear transformation

$$H : \mathbb{P}_2 \to M_{2\times 2}$$

given by

$$H(a+bx+cx^2) = \begin{bmatrix} a+b & a-b \\ c & c-a \end{bmatrix}$$

We find the matrix of H with respect to the standard basis $\mathcal{C} = \{1, x, x^2\}$ for $\mathbb{P}_2$ and
$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for $M_{2\times 2}$.

- STEP 1 We find the effect of H on each of the basis elements:

$$H(1) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad H(x) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, H(x^2) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

- STEP 2 The corresponding coordinate vectors are

$$[H(1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad [H(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, [H(x^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

- STEP 3 Hence the matrix for H relative to the bases $\mathcal{C}$ and $\mathcal{B}$ is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Linear transformations from V to V

The most common case is when $T : V \to V$ and $\mathcal{B} = \mathcal{C}$. In this case $\underset{\mathcal{B}\leftarrow\mathcal{B}}{T}$ is written $[T]_{\mathcal{B}}$ and is the *matrix for* T *relative to* $\mathcal{B}$ or simply the *$\mathcal{B}$-matrix of* T.
The $\mathcal{B}$-matrix for $T : V \to V$ satisfies

$$[T(\mathbf{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}, \quad \text{for all } \mathbf{x} \in V. \tag{1}$$

$$\begin{array}{ccc} \mathbf{x} & \xrightarrow{\quad T \quad} & T(\mathbf{x}) \\ \downarrow & & \downarrow \\ [\mathbf{x}]_{\mathcal{B}} & \xrightarrow{\text{multiplication by } [T]_{\mathcal{B}}} & [T(\mathbf{x})]_{\mathcal{B}} \end{array}$$

Examples

Example 9

Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by

$$T(p(x)) = p(2x - 1).$$

We find the matrix of T with respect to $\mathcal{E} = \{1, x, x^2\}$

- STEP 1 It is clear that

$$T(1) = 1, \quad T(x) = 2x - 1,$$
$$T(x^2) = (2x - 1)^2 = 1 - 4x + 4x^2$$

- STEP 2 So the coordinate vectors are

$$[T(1)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [T(x)]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \left[T(x^2)\right]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}.$$

- STEP 3 Therefore

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

Example 10

We compute $T(3 + 2x - x^2)$ using part (a).

The coordinate vector of $p(x) = (3 + 2x - x^2)$ with respect to $\mathcal{E}$ is given by

$$[p(x)]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}.$$

We use the relationship

$$[T(p(x))]_{\mathcal{E}} = [T]_{\mathcal{E}}[p(x)]_{\mathcal{E}}.$$

This gives

$$\begin{aligned}[T(3+2x-x^2)]_{\mathcal{E}} &= [T(p(x))]_{\mathcal{E}} \\ &= [T]_{\mathcal{E}}[p(x)]_{\mathcal{E}} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix}\end{aligned}$$

It follows that $T(3+2x-x^2) = 8x - 4x^2$.

Example 11

Consider the linear transformation $F : M_{2\times 2} \to M_{2\times 2}$ given by

$$F(A) = A + A^T$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

We use the basis

$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for $M_{2\times 2}$ to find a matrix representation for T.

More explicitly F is given by

$$F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

- STEP 1 We find the effect of F on each of the basis elements:

$$F\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, F\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$F\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, F\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

• STEP 2 The corresponding coordinate vectors are

$$\left[F\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix}2\\0\\0\\0\end{bmatrix}, \left[F\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix},$$

$$\left[F\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \left[F\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix}0\\0\\0\\2\end{bmatrix}.$$

• STEP 3 Hence the matrix representing F is

$$\begin{bmatrix}2 & 0 & 0 & 0\\0 & 1 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 2\end{bmatrix}.$$

Example 12

Let $V = \text{Span}\,\{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\}$.
We find the matrix of the differential operator D with respect to

$$\mathcal{B} = \{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\}.$$

• STEP 1 We see that

$$\begin{aligned}D(e^{2x}) &= 2e^{2x}\\ D(e^{2x}\cos x) &= 2e^{2x}\cos x - e^{2x}\sin x\\ D(e^{2x}\sin x) &= 2e^{2x}\sin x + e^{2x}\cos x\end{aligned}$$

- STEP 2 So the coordinate vectors are

$$[D(e^{2x})]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, [D(e^{2x}\cos x)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix},$$

$$\text{and} \qquad [D(e^{2x}\sin x)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

- STEP 3 Hence

$$[D]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Example 13

We use this result to find the derivative of $f(x) = 3e^{2x} - e^{2x}\cos x + 2e^{2x}\sin x$.

The coordinate vector of $f(x)$ is given by

$$[f]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

We do this calculation using

$$[D(f)]_{\mathcal{B}} = [D]_{\mathcal{B}}[f]_{\mathcal{B}}.$$

This gives

$$\begin{aligned} [D(f)]_{\mathcal{B}} &= [D]_{\mathcal{B}}[f]_{\mathcal{B}} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix}. \end{aligned}$$

This indicates that

$$f'(x) = 6e^{2x} + 5e^{2x}\sin x.$$

You should check this result by differentiation.

Example 14

We use the previous result to find $\int (4e^{2x} - 3e^{2x} \sin x)\ dx$

We recall that with the basis $\mathcal{B} = \{e^{2x}, e^{2x} \cos x, e^{2x} \sin x\}$ the matrix representation of the differential operator D is given by

$$[D]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}.$$

We also notice that $[D]_{\mathcal{B}}$ is invertible with inverse:

$$[D]_{\mathcal{B}}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 2/5 & -1/5 \\ 0 & 1/5 & 2/5 \end{bmatrix}.$$

The coordinate vector of $4e^{2x} - 3e^{2x} \sin x$ with respect to the basis $\mathcal{B}$ is given by $\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$. We use this together with the inverse of $[D]_{\mathcal{B}}$ to find the antiderivative $\int (4e^{2x} - 3e^{2x} \sin x)\ dx$:

$$[D]_{\mathcal{B}}^{-1}[4e^{2x} - 3e^{2x}]_{\mathcal{B}} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 2/5 & -1/5 \\ 0 & 1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3/5 \\ -6/5 \end{bmatrix}.$$

So the antiderivative of $4e^{2x} - 3e^{2x}$ in the vector space V is $2e^{2x} + \frac{3}{5}e^{2x} \cos x - \frac{6}{5}e^{2x} \sin x$, and we can deduce that $\int (4e^{2x} - 3e^{2x} \sin x)\ dx = 2e^{2x} + \frac{3}{5}e^{2x} \cos x - \frac{6}{5}e^{2x} \sin x + C$ where C denotes a constant.

Linear transformations and diagonalisation

In an applied problem involving $\mathbb{R}^n$, a linear transformation T usually appears as a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$. If A is diagonalisable, then there is a basis $\mathcal{B}$ for $\mathbb{R}^n$ consisting of eigenvectors of A. In this case the $\mathcal{B}$-matrix for T is diagonal, and diagonalising A amounts to finding a diagonal matrix representation of $\mathbf{x} \mapsto A\mathbf{x}$.

Theorem

Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed by the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Proof.

Denote the columns of P by $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$, so that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ and

$$P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}.$$

In this case, P is the change of coordinates matrix $P_{\mathcal{B}}$ where

$$P[\mathbf{x}]_{\mathcal{B}} = \mathbf{x} \quad \text{and} \quad [\mathbf{x}]_{\mathcal{B}} = P^{-1}\mathbf{x}.$$

If T is defined by $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^n$, then

$$\begin{aligned}
[T]_{\mathcal{B}} &= \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{B}} & \cdots & T(\mathbf{b}_n)]_{\mathcal{B}} \end{bmatrix} \\
&= \begin{bmatrix} [A\mathbf{b}_1]_{\mathcal{B}} & \cdots & [A\mathbf{b}_n]_{\mathcal{B}} \end{bmatrix} \\
&= \begin{bmatrix} P^{-1}A\mathbf{b}_1 & \cdots & P^{-1}A\mathbf{b}_n \end{bmatrix} \\
&= P^{-1}A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} \\
&= P^{-1}AP = D
\end{aligned}$$

□

In the proof of the previous theorem the fact that D is diagonal is never used. In fact the following more general result holds:

If an $n \times n$ matrix A is similar to a matrix C with $A = PCP^{-1}$, then C is the $\mathcal{B}$-matrix of the transformation $\mathbf{x} \to A\mathbf{x}$ where $\mathcal{B}$ is the basis of $\mathbb{R}^n$ formed by the columns of P.

Example

Example 15

Consider the matrix $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$. T is the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$. We find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

The first step is to find the eigenvalues and corresponding eigenspaces for A:

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} 4-\lambda & -2 \\ -1 & 3-\lambda \end{bmatrix} \\
&= (4-\lambda)(3-\lambda) - 2 \\
&= \lambda^2 - 7\lambda + 10 \\
&= (\lambda - 2)(\lambda - 5).
\end{aligned}$$

The eigenvalues of A are $\lambda = 2$ and $\lambda = 5$. We need to find a basis vector for each of these eigenspaces.

$$\begin{aligned} E_2 &= \text{Nul} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \\ &= \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \\ E_5 &= \text{Nul} \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix} \\ &= \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Put $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

Then $[T]_{\mathcal{B}} = D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, and with $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ and $P^{-1}AP = D$, or equivalently, $A = PDP^{-1}$.

Overview

We've looked at eigenvalues and eigenvectors from several perspectives, studying how to find them and what they tell you about the linear transformation associated to a matrix.

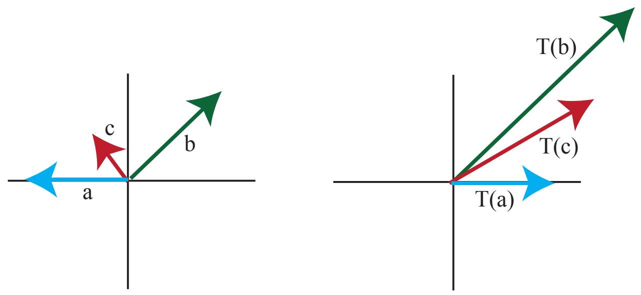
Question

What happens when the characteristic equation has complex roots?

From Lay, §5.5

Warm-up unquiz for review

Suppose that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ acts as shown in the picture:



Write a matrix for T with respect to a basis of your choice.

Existence of Complex Eigenvalues

Since the characteristic equation of an $n \times n$ matrix involves a polynomial of degree n, there will be times when the roots of the characteristic equation will be complex. Thus, even if we start out considering matrices with real entries, we're naturally lead to consider complex numbers.

We'll focus on understanding what **complex** eigenvalues mean when **the entries of the matrix with which we are working are all real numbers**. For simplicity, we'll restrict to the case of 2×2 matrices.

Example 1

Let $A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ for some real φ. The roots of the characteristic equation are $\cos\varphi \pm i\sin\varphi$.

What does the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$ (for all $\mathbf{x} \in \mathbb{R}^2$) do to vectors in $\mathbb{R}^2$?

Since the i^{th} column of the matrix is $T(\mathbf{e_i})$, we see that the linear transformation T_A is the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle.

A rotation in $\mathbb{R}^2$ cannot have a real eigenvector unless $\varphi = 2k\pi$ or $\varphi = \pi + 2k\pi$ for $k \in \mathbb{Z}$!

What about (complex) eigenvectors for such an A?

Let's take $\varphi = \pi/3$, so that multiplication by A corresponds to a rotation through $\pi/3$ (60^0). Then we get

$$A = \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

What happens when we try to find eigenvalues and eigenvectors?

The characteristic polynomial of A is

$$(1/2 - \lambda)^2 + (\sqrt{3}/2)^2 = \lambda^2 - \lambda + 1$$

and the eigenvalues are

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

Take $\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. We find the eigenvectors in the usual way by solving $(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$.

$$A - \lambda_1 I = \begin{bmatrix} -i\sqrt{3}/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -i\sqrt{3}/2 \end{bmatrix} \to \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}.$$

We solve the associated equation as usual, so we see that $ix + y = 0$.

Thus one possible eigenvector is $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

(All the other associated eigenvectors are of the form $\alpha\mathbf{x}_1 = \begin{bmatrix} \alpha \\ -i\alpha \end{bmatrix}$, where α is any non-zero number in $\mathbb{C}$.)

For $\lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ we get $\mathbf{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ as an associated complex eigenvector.

(All the other associated eigenvectors are of the form $\alpha\mathbf{x}_2 = \begin{bmatrix} \alpha \\ i\alpha \end{bmatrix}$, where α is any non-zero number in $\mathbb{C}$.)

We can check that these two vectors are in fact eigenvectors:

$$\begin{aligned} A\mathbf{x}_1 &= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= \begin{bmatrix} 1/2 + i\sqrt{3}/2 \\ \sqrt{3}/2 - i/2 \end{bmatrix} \\ &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) \begin{bmatrix} 1 \\ -i \end{bmatrix}. \end{aligned}$$

Similarly,

$$A\mathbf{x}_2 = \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Example 2

Find the eigenvectors associated to the matrix $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

The characteristic polynomial is

$$\det \begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda) + 2 = \lambda^2 - 8\lambda + 17.$$

The roots are

$$\lambda = \frac{8 \pm \sqrt{64-68}}{2} = \frac{8 \pm \sqrt{-4}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i.$$

Since complex roots always come in conjugate pairs, it follows that if $a + bi$ is an eigenvalue for A, then $a - bi$ will also be an eigenvalue for A.

Take $\lambda_1 = 4 + i$. We find a corresponding eigenvector:

$$A - \lambda_1 I = \begin{bmatrix} 5-(4+i) & -2 \\ 1 & 3-(4+i) \end{bmatrix} = \begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}$$

Row reduction of the usual augmented matrix is quite unpleasant by hand because of the complex numbers.

However, there is an observation that simplifies matters: Since $4 + i$ is an eigenvalue, the system of equations

$$\begin{aligned} (1-i)x_1 \; - \; 2x_2 &= 0 \\ x_1 \; - \; (1+i)x_2 &= 0 \end{aligned}$$

has a non trivial solution.

Therefore both equations determine the same relationship between x_1 and x_2, and either equation can be used to express one variable in terms of the other.

As these two equations both give the same information, we can use the second equation. It gives

$$x_1 = (1+i)x_2,$$

where x_2 is a free variable. If we take $x_2 = 1$, we get $x_1 = 1+i$ and hence an eigenvector is $\mathbf{x}_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$.

If we take $\lambda_2 = 4 - i$, and proceed as for λ_1 we get that $\mathbf{x}_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ is a corresponding eigenvector.

Just as the eigenvalues come in a pair of complex conjugates, and so do the eigenvectors.

Normal form

When a matrix is diagonalisable, it's similar to a diagonal matrix: $A = PDP^{-1}$.
It's also similar to many other matrices, but we think of the diagonal matrix as the "best" representative of the class, in the sense that it expresses the associated linear transformation with respect to a most natural basis (i.e., a basis of eigenvectors.)
Of course, not all matrices are diagonalisable, so today we consider the following question:

Question

Given an arbitrary matrix, is there a "best" representative of its similarity class?

"Best" isn't a precise term, but let's interpret this as asking whether there's some basis for which the action of the associate linear transformation is most transparent.

Example 3

Consider the matrix $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$.

The characteristic polynomial is $1 - \lambda^3$, with roots $1, -1 \pm i\frac{\sqrt{3}}{2}$, the three cube roots of unity in $\mathbb{C}$.
A choice of corresponding eigenvectors is, for example,

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 + i\frac{\sqrt{3}}{2} \\ 1 + i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -1 - i\frac{\sqrt{3}}{2} \\ 1 - i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}.$$

Notice that we have one real eigenvector corresponding to the real eigenvalue 1, and two complex eigenvectors corresponding to the complex eigenvalues. Notice that also in this case the complex eigenvalues and eigenvectors come in pairs of conjugates.

Advantages of complex linear algebra

Doing computations by hand is messier when we work over $\mathbb{C}$, but much of the theory is cleaner! When the scalars are complex, rather than real

- matrices always have eigenvalues and eigenvectors; and
- every linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ can be represented by an upper triangular matrix.

We don't have time to explore the implications fully, but we can take a quick look at some of the interesting structure that emerges immediately.

A real matrix acting on $\mathbb{C}$

Eigenvalues come in conjugate pairs.
If A is an $m \times n$ matrix with entries in $\mathbb{C}$, then $\bar{A}$ denotes the matrix whose entries are the complex conjugates of the entries in A.
Let A be an $n \times n$ matrix whose entries are real. Then $\overline{A} = A$. So

$$\overline{A\mathbf{x}} = \overline{A}\overline{\mathbf{x}} = A\overline{\mathbf{x}}$$

for any vector $\mathbf{x} \in \mathbb{C}^n$.
If λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector in $\mathbb{C}^n$, then

$$A\overline{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}.$$

This shows that $\overline{\lambda}$ is also an eigenvalue of A with $\overline{\mathbf{x}}$ a corresponding eigenvector.
So...
...when A is a real matrix, its complex eigenvalues occur in conjugate pairs.

Some special 2×2 matrices

Consider the matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are real numbers and not both 0.

$$C - I\lambda = \begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix},$$

so the characteristic equation for C is

$$0 = (a-\lambda)^2 + b^2 = \lambda^2 - 2a\lambda + a^2 + b^2.$$

Using the quadratic formula, the eigenvalues of C are

$$\lambda = a \pm bi.$$

So if $b \neq 0$, the eigenvalues are not real.

Notice that this generalises our earlier observation about rotation matrices.
In fact...

...apply some magic...

If we now take $r = |\lambda| = \sqrt{a^2 + b^2}$ then we can write

$$C = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

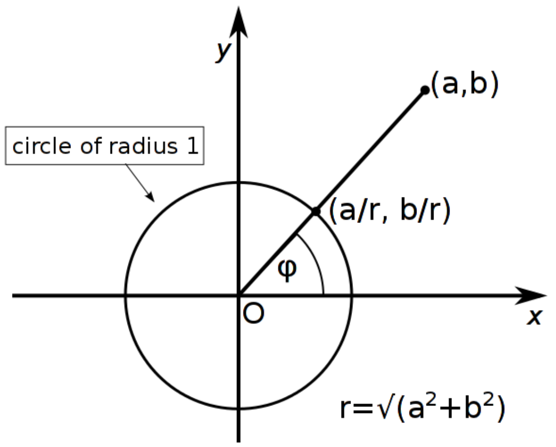
where φ is the angle between the positive x-axis and the ray from $(0,0)$ through (a,b). Here we used the fact that

$$\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2+b^2}{r^2} = \frac{r^2}{r^2} = 1\,.$$

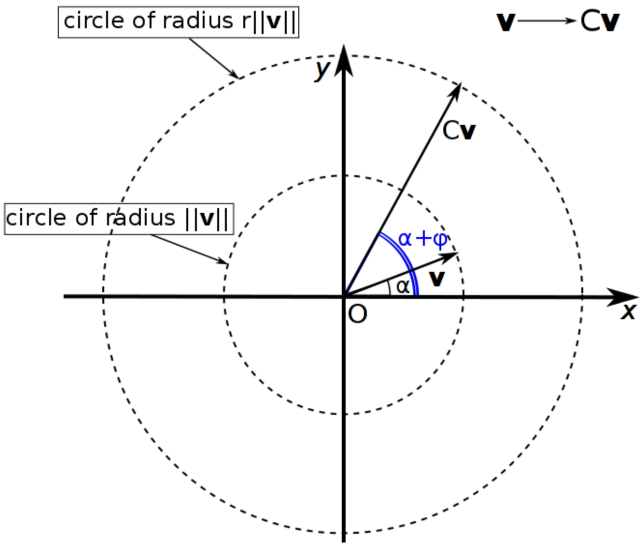
Thus the point $(a/r, b/r)$ lies on the circle of radius 1 with center at the origin and a/r, b/r can be seen as the cosine and sine of the angle between the positive x-axis and the ray from $(0,0)$ through $(a/r, b/r)$ (which is the same as the angle between the positive x-axis and the ray from $(0,0)$ through (a,b)).

The transformation $\mathbf{x} \mapsto C\mathbf{x}$ may be viewed as the composition of a rotation through the angle φ and a scaling by $r = |\lambda|$.

The angle φ



The action of C



Example 4

What is the geometric action of $C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{R}^2$?

From what we've just seen, C has eigenvalues $\lambda = 1 \pm i$, so $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. We can therefore rewrite C as

$$C = \sqrt{2} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \sqrt{2} \begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix}.$$

So C acts as a rotation through $\pi/4$ together with a multiplication by $\sqrt{2}$.

To verify this, we look at the repeated action of C on a point $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. (Note $|\mathbf{x}_0| = 1$.)

$$\mathbf{x}_1 = C\mathbf{x}_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, ||\mathbf{x}_1|| = \sqrt{2},$$

$$\mathbf{x}_2 = C\mathbf{x}_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, ||\mathbf{x}_2|| = 2,$$

$$\mathbf{x}_3 = C\mathbf{x}_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, ||\mathbf{x}_3|| = 2\sqrt{2}, \ldots$$

If we continue, we'll find a spiral of points each one further away from $(0, 0)$ than the previous one.

Real and imaginary parts of vectors

The *complex conjugate* of a complex vector $\mathbf{x}$ in $\mathbb{C}^n$ is the vector $\bar{\mathbf{x}}$ in $\mathbb{C}^n$ whose entries are the complex conjugates of the entries in $\mathbf{x}$.

The *real* and *imaginary parts* of a complex vector $\mathbf{x}$ are the vectors Re $\mathbf{x}$ and Im $\mathbf{x}$ formed from the real and imaginary parts of the entries of $\mathbf{x}$.

If $\mathbf{x} = \begin{bmatrix} 1+2i \\ -3i \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + i \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$, then

$$\text{Re } \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \text{Im } \mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \text{ and}$$

$$\bar{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} - i \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-2i \\ 3i \\ 5 \end{bmatrix}.$$

We'll use this idea in the next example.

The rotation hidden in a real matrix with a complex eigenvalue

Example 5

Show that $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$ is similar to a matrix of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

The characteristic polynomial of A is

$$\det \begin{bmatrix} 2-\lambda & 1 \\ -2 & -\lambda \end{bmatrix} = (2-\lambda)(-\lambda) + 2 = \lambda^2 - 2\lambda + 2.$$

So A has complex eigenvalues

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Take $\lambda_1 = 1 - i$. To find a corresponding eigenvector we find $A - \lambda_1 I$:

$$A - \lambda_1 I = \begin{bmatrix} 2-(1-i) & 1 \\ -2 & 0-(1-i) \end{bmatrix} = \begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix}$$

We can use the first row of the matrix to solve $(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$:

$$(1+i)x_i + x_2 = 0 \quad \text{or} \quad x_2 = -(1+i)x_1.$$

If we take $x_1 = 1$ we get an eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$$

We now construct a real 2×2 matrix P:

$$P = \begin{bmatrix} \text{Re } \mathbf{v}_1 & \text{Im } \mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

We have not justified why we would try this!

Note that $P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$.

Then calculate

$$\begin{aligned} C &= P^{-1}AP \\ &= \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

We recognise this matrix, from the previous example, as the composition of a counterclockwise rotation by $\pi/4$ and a scaling by $\sqrt{2}$. This is the rotation "inside" A. We can write A:

$$A = PCP^{-1} = P\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}P^{-1}.$$

From the last lecture, we know that C is the matrix of the linear transformation $\mathbf{x} \to A\mathbf{x}$ relative to the basis $\mathcal{B} = \left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}\right\}$ formed by the columns of P.

This shows that when we represent the transformation in terms of the basis $\mathcal{B}$, the transformation $\mathbf{x} \to A\mathbf{x}$ "looks like" the composition of a scaling and a rotation. As promised, using a non-standard basis we can sometimes uncover the hidden geometric properties of a linear transformation!

Example 6

Consider the matrix $A = \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$.

The characteristic polynomial of A is given by

$$\det\begin{bmatrix}1-\lambda & -1\\ 1 & -\lambda\end{bmatrix} = (1-\lambda)(-\lambda) + 1 = \lambda^2 - \lambda + 1.$$

This is the same polynomial as for the matrix in Example 1. So we know that A has complex eigenvalues and therefore complex eigenvectors.

To see how multiplication by A affects points, take an arbitrary point, say $\mathbf{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}$, and then plot successive images of this point under repeated multiplication by A.

The first few points are

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix},$$

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix},$$

$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix},$$

$$\mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}, \ldots$$

You could try this also for matrices $\begin{bmatrix}0.1 & -0.2\\ 0.1 & 0.3\end{bmatrix}$ and $\begin{bmatrix}2 & 1\\ -2 & 0\end{bmatrix}$.

The theorem (and why it's true)

Theorem

Let A be a 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = PCP^{-1}, \text{ where } P = \begin{bmatrix} Re\,\mathbf{v} & Im\,\mathbf{v} \end{bmatrix}$$

$$\text{and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Sketch of proof

Suppose that A is a real 2×2 matrix, with a complex eigenvalue $\lambda = a - ib$, $b \neq 0$, and a corresponding complex eigenvector $\mathbf{v} = \mathbf{v}_1 + i\mathbf{v}_2$ where $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$. Then

- $\mathbf{v}_2 \neq \mathbf{0}$ because otherwise $A\mathbf{v} = A\mathbf{v}_1$ would be real, whereas $\lambda\mathbf{v} = \lambda\mathbf{v}_1$ is not.
- If $\mathbf{v}_1 = \alpha\mathbf{v}_2$, for some (necessarily real) α,

$$A(\mathbf{v}) = A((\alpha + i)\mathbf{v}_2) = (\alpha + i)A\mathbf{v}_2 = (\alpha + i)\lambda\mathbf{v}_2$$

  whence the real vector $A\mathbf{v}_2$ equals $\lambda\mathbf{v}_2$ which is not real.

Thus the real vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, and give a basis for $\mathbb{R}^2$.

Equate the real and imaginary parts in the two formulas

$$A\mathbf{v} = (a - ib)\mathbf{v} = (a - ib)(\mathbf{v}_1 + i\mathbf{v}_2) = (a\mathbf{v}_1 + b\mathbf{v}_2) + i(a\mathbf{v}_2 - b\mathbf{v}_1),$$

and

$$A\mathbf{v} = A(\mathbf{v}_1 + i\mathbf{v}_2) = A\mathbf{v}_1 + iA\mathbf{v}_2.$$

This gives $A\mathbf{v}_1 = a\mathbf{v}_1 + b\mathbf{v}_2$ and $A\mathbf{v}_2 = a\mathbf{v}_2 - b\mathbf{v}_1$ so that

$$\begin{aligned} A\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} &= \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 \end{bmatrix} \\ &= \begin{bmatrix} a\mathbf{v}_1 + b\mathbf{v}_2 & a\mathbf{v}_2 - b\mathbf{v}_1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \end{aligned}$$

So with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$, the transformation T_A has matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Setting $\sin\varphi = \dfrac{b}{\sqrt{a^2+b^2}}, \cos\varphi = \dfrac{a}{\sqrt{a^2+b^2}}$,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix},$$

which is a scaling and rotation. And all of this is determined by the complex eigenvalue $a - ib$.

Of course, if $a - ib$ is an eigenvalue with eigenvector $\mathbf{v}_1 + i\mathbf{v}_2$, $a + ib$ is an eigenvalue, with eigenvector $\mathbf{v}_1 - i\mathbf{v}_2$.

Example 7

What is the geometric action of $A = \begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$ on $\mathbb{R}^2$?

As a first step we find the eigenvalues and eigenvectors associated with A.

$$\begin{aligned} \det(A - \lambda I) &= \begin{bmatrix} -5-\lambda & -5 \\ 5 & -5-\lambda \end{bmatrix} \\ &= (-5-\lambda)^2 + 25 \\ &= \lambda^2 + 10\lambda + 50 \end{aligned}$$

This gives

$$\lambda = \frac{-10 \pm \sqrt{100-200}}{2} = \frac{-10 \pm 10i}{2} = -5 \pm 5i.$$

Consider the eigenvalue $\lambda = -5 - 5i$. We will find the corresponding eigenspace:

$$\begin{aligned} E_\lambda &= \text{Nul } (A - \lambda I) \\ &= \text{Nul} \begin{bmatrix} 5i & -5 \\ 5 & 5i \end{bmatrix} \\ &= \text{Span } \left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \right\} \end{aligned}$$

where Span here stands for *complex* span, that is the set of all scalar multiples $\alpha \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \alpha \\ i\alpha \end{bmatrix}$ of $\begin{bmatrix} 1 \\ i \end{bmatrix}$, where α is in $\mathbb{C}$.

Choosing $\begin{bmatrix} 1 \\ i \end{bmatrix}$ as our eigenvector we find the associated matrices P and C:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}.$$

It is easy to check that

$$A = PCP^{-1} \text{ or equivalently } AP = PC.$$

Further

$$C = \begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix} = 5\sqrt{2} \begin{bmatrix} -1/\sqrt{2} & -1\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

The scaling factor is $5\sqrt{2}$. The angle of rotation is given by $\cos\varphi = -1/\sqrt{2}, \sin\varphi = 1/\sqrt{2}$, which gives $\phi = 3\pi/4$ (135°).

Overview

Yesterday we studied how real 2×2 matrices act on $\mathbb{C}$. Just as the action of a diagonal matrix on $\mathbb{R}^2$ is easy to understand (i.e., scaling each of the basis vectors by the corresponding diagonal entry), the action of a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ determines a composition of rotation and scaling.
We also saw that any 2×2 matrix with complex eigenvalues is similar to such a "standard" form.
Today we'll return to the study of matrices with real eigenvalues, using them to model discrete dynamical systems.

From Lay, §5.6

The main ideas

In this section we will look at discrete linear dynamical systems. *Dynamics* describe the evolution of a system over time, and a *discrete* system is one where we sample the state of the system at intervals of time, as opposed to studying its continuous behaviour. Finally, these systems are *linear* because the change from one state to another is described by a vector equation like

$$(*) \qquad \mathbf{x}_{k+1} = A\mathbf{x}_k \,.$$

where A is an $n \times n$ matrix and the $\mathbf{x}_k$'s are vectors $\mathbb{R}^n$.

You should look at the equation above as a recursive relation. Given an initial vector $\mathbf{x}_0$ we obtain a sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, ..$ where for every k the vector $\mathbf{x}_{k+1}$ is obtained from the previous vector $\mathbf{x}_k$ using the relation $(*)$. We are generally interested in the long term behaviour of such a system.

The applications in Lay focus on ecological problems, but also apply to problems in physics, engineering and many other scientific fields.

Initial assumptions
We'll start by describing the circumstances under which our techniques will be effective:

- The matrix A is diagonalisable.
- A has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.
- The eigenvectors are arranged so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$, any initial vector $\mathbf{x}_0$ can be written

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

This eigenvector decomposition of $\mathbf{x}_0$ determines what happens to the sequence $\{\mathbf{x}_k\}$.

Since

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n,$$

we have

$$\begin{aligned} \mathbf{x}_1 = A\mathbf{x}_0 &= c_1 A\mathbf{v}_1 + \cdots + c_n A\mathbf{v}_n \\ &= c_1\lambda_1\mathbf{v}_1 + \cdots + c_n\lambda_n\mathbf{v}_n \\ \mathbf{x}_2 = A\mathbf{x}_1 &= c_1\lambda_1 A\mathbf{v}_1 + \cdots + c_n\lambda_n A\mathbf{v}_n \\ &= c_1(\lambda_1)^2\mathbf{v}_1 + \cdots + c_n(\lambda_n)^2\mathbf{v}_n \end{aligned}$$

and in general,

$$\mathbf{x}_k = c_1(\lambda_1)^k\mathbf{v}_1 + \cdots + c_n(\lambda_n)^k\mathbf{v}_n \tag{1}$$

We are interested in what happens as $k \to \infty$.

Predator - Prey Systems

Example

See Example 1, Section 5.6

The owl and wood rat populations at time k are described by $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, where k is the time in months, O_k is the number of owls in the region studied, and R_k is the number of rats (measured in thousands). Since owls eat rats, we should expect the population of each species to affect the future population of the other one.

The changes in theses populations can be described by the equations:

$$\begin{aligned} O_{k+1} &= (0.5)O_k + (0.4)R_k \\ R_{k+1} &= -p \cdot O_k + (1.1)R_k \end{aligned}$$

where p is a positive parameter to be specified.

In matrix form this is

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0.5 & 0.4 \\ -p & 1.1 \end{bmatrix}\mathbf{x}_k.$$

Example (Case 1)

$p = 0.104$

This gives $A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$

According to the book, the eigenvalues for A are $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$. Corresponding eigenvectors are, for example,

$$\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

An initial population $\mathbf{x}_0$ can be written as $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then for $k \geq 0$,

$$\begin{aligned}\mathbf{x}_k &= c_1(1.02)^k\mathbf{v}_1 + c_2(0.58)^k\mathbf{v}_2 \\ &= c_1(1.02)^k\begin{bmatrix}10\\13\end{bmatrix} + c_2(0.58)^k\begin{bmatrix}5\\1\end{bmatrix}\end{aligned}$$

As $k \to \infty$, $(0.58)^k \to 0$. Assume $c_1 > 0$. Then for large k,

$$x_k \approx c_1(1.02)^k\begin{bmatrix}10\\13\end{bmatrix}$$

and

$$\mathbf{x}_{k+1} \approx c_1(1.02)^{k+1}\begin{bmatrix}10\\13\end{bmatrix} \approx 1.02\mathbf{x}_k.$$

The last approximation says that eventually both the population of rats and the population of owls grow by a factor of almost 1.02 per month, a 2% growth rate.

The ratio 10 to 13 of the entries in $\mathbf{x}_k$ remain the same, so for every 10 owls there are 13 thousand rats.

This example illustrates some general facts about a dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ when

- $|\lambda_1| \geq 1$ and
- $1 > |\lambda_j|$ for $j \geq 2$ and
- $\mathbf{v}_1$ is an eigenvector associated with λ_1.

If $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, with $c_1 \neq 0$, then for all sufficiently large k,

$$\mathbf{x}_{k+1} \approx \lambda_1\mathbf{x}_k \quad \text{and} \quad \mathbf{x}_k \approx c_1(\lambda)^k\mathbf{v}_1.$$

Example (Case 2)

We consider the same system when $p = 0.2$ (so the predation rate is higher than in the previous Example (1), where we had taken $p = 0.104 < 0.2$). In this case the matrix A is

$$\begin{bmatrix}0.5 & 0.4\\-0.2 & 1.1\end{bmatrix}.$$

Here

$$A - \lambda I = \begin{bmatrix}0.5-\lambda & 0.4\\-0.2 & 1.1-\lambda\end{bmatrix}$$

and the characteristic equation is

$$\begin{aligned}0 &= (0.5-\lambda)(1.1-\lambda) + (0.4)(0.2) \\ &= 0.55 - 1.6\lambda + \lambda^2 + 0.08 \\ &= \lambda^2 - 1.6\lambda + 0.63 \\ &= (\lambda - 0.9)(\lambda - 0.7)\end{aligned}$$

When $\lambda = 0.9$,

$$E_{0.9} = \text{Nul} \begin{bmatrix} -0.4 & 0.4 \\ -0.2 & 0.2 \end{bmatrix} \to \text{Nul} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

and an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

When $\lambda = 0.7$

$$E_{0.7} = \text{Nul} \begin{bmatrix} -0.2 & 0.4 \\ -0.2 & 0.4 \end{bmatrix} \to \text{Nul} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

and an eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

This gives

$$\mathbf{x}_k = c_1(0.9)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(0.7)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} \to \mathbf{0},$$

as $k \to \infty$.

The higher predation rate cuts down the owls' food supply, and in the long term both populations die out.

Example (Case 3)

We consider the same system again when $p = 0.125$. In this case the matrix A is

$$\begin{bmatrix} 0.5 & 0.4 \\ -0.125 & 1.1 \end{bmatrix}.$$

Hence

$$A - \lambda I = \begin{bmatrix} 0.5 - \lambda & 0.4 \\ -0.125 & 1.1 - \lambda \end{bmatrix}$$

and the characteristic equation is

$$\begin{aligned} 0 &= (0.5 - \lambda)(1.1 - \lambda) + (0.4)(0.125) \\ &= 0.55 - 1.6\lambda + \lambda^2 + 0.05 \\ &= \lambda^2 - 1.6\lambda + 0.6 \\ &= (\lambda - 1)(\lambda - 0.6). \end{aligned}$$

When $\lambda = 1$,

$$E_1 = \text{Nul}\begin{bmatrix} -0.5 & 0.4 \\ -0.125 & 0.1 \end{bmatrix} \to \text{Nul}\begin{bmatrix} 1 & -0.8 \\ 0 & 0 \end{bmatrix}$$

and an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}$.

When $\lambda = 0.6$

$$E_{0.6} = \text{Nul}\begin{bmatrix} -0.1 & 0.4 \\ -0.125 & 0.5 \end{bmatrix} \to \text{Nul}\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

and an eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

This gives

$$\mathbf{x}_k = c_1(1)^k \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} + c_2(0.6)^k \begin{bmatrix} 4 \\ 1 \end{bmatrix} \to c_1 \begin{bmatrix} 0.8 \\ 1 \end{bmatrix},$$

as $k \to \infty$.

In this case the population reaches an equilibrium, where for every 8 owls there are 10 thousand rats. The size of the population depends only on the values of c_1.

This equilibrium is not considered stable as small changes in the birth rates or the predation rate can change the situation.

Graphical Description of Solutions

When A is a 2×2 matrix we can describe the evolution of a dynamical system geometrically.

The equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ determines an infinite collection of equations. Beginning with an initial vector $\mathbf{x}_0$, we have

$$\begin{aligned} \mathbf{x}_1 &= A\mathbf{x}_0 \\ \mathbf{x}_2 &= A\mathbf{x}_1 \\ \mathbf{x}_3 &= A\mathbf{x}_2 \\ &\vdots \end{aligned}$$

The set $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots\}$ is called a trajectory of the system.

Note that $\mathbf{x}_k = A^k\mathbf{x}_0$.

Examples

Example 1

Let $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}$. Plot the first five points in the trajectories with the following initial vectors:

$$(a)\ \mathbf{x}_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \quad (b)\ \mathbf{x}_0 = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

$$(c)\ \mathbf{x}_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad (d)\ \mathbf{x}_0 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Notice that since A is already diagonal, the computations are much easier!

(a) For $\mathbf{x}_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}$, we compute

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix} \qquad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 1.25 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 0.625 \\ 0 \end{bmatrix} \qquad \mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} 0.3125 \\ 0 \end{bmatrix}$$

These points converge to the origin along the x-axis.

(Note that $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for the matrix).

(b) The situation is similar for the case $\mathbf{x}_0 = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$, except that the convergence is along the y-axis.

(c) For the case $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, we get

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 2 \\ 3.2 \end{bmatrix} \qquad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2.56 \end{bmatrix}$$

$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 0.5 \\ 2.048 \end{bmatrix} \qquad \mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} 0.25 \\ 1.6384 \end{bmatrix}$$

These points also converge to the origin, but not along a direct line. The trajectory is an arc that gets closer to the y-axis as it converges to the origin.

The situation is similar for case (d) with convergence also toward the y-axis.

In this example every trajectory converges to $\mathbf{0}$. The origin is called an attractor for the system.

We can understand why this happens when we consider the eigenvalues of A: 0.8 and 0.5. These have corresponding eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
So, for an initial vector

$$\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we have

$$\mathbf{x}_k = A^k \mathbf{x}_0 = c_1 (0.8)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 (0.5)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Because both $(0.8)^k$ and $(0.5)^k$ approach zero as k gets large, $\mathbf{x}_k$ approaches $\mathbf{0}$ for any initial vector $\mathbf{x}_0$.
Because $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to the larger eigenvalue of A, $\mathbf{x}_k$ approaches a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as long as $c_1 \neq 0$.

Graphical example

Dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} .80 & 0 \\ 0 & .64 \end{bmatrix}$$

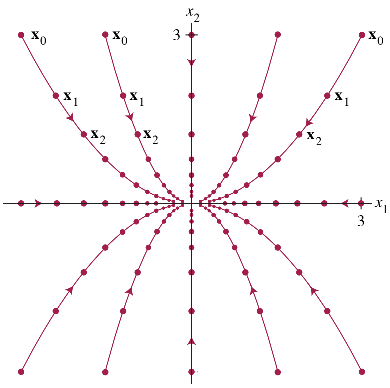


FIGURE 1 The origin as an attractor.

Example 2

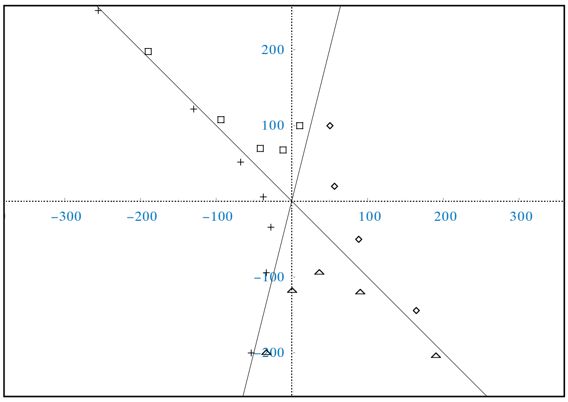
Describe the trajectories of the dynamical system associated to the matrix $A = \begin{bmatrix} 1.7 & -0.3 \\ -1.2 & 0.8 \end{bmatrix}$.

The eigenvalues of A are 2 and 0.5, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.
As above, the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ has solution

$$\mathbf{x}_k = 2^k c_1 \mathbf{v}_1 + (.05)^k c_2 \mathbf{v}_2$$

where c_1, c_2 are determined by $\mathbf{x}_0$.
Thus for $\mathbf{x}_0 = \mathbf{v}_1$, $\mathbf{x}_k = 2^k \mathbf{v}_1$, and this is unbounded for large k, whereas for $\mathbf{x}_0 = \mathbf{v}_2$, $\mathbf{x}_k = (0.5)^k \mathbf{v}_2 \to \mathbf{0}$.
In this example we see different behaviour in different directions. We describe this by saying that the origin is a *saddle point*.

Here are some trajectories with different starting points:



saddle

If a starting point is closer to $\mathbf{v}_2$ it is initially attracted to the origin, and when it gets closer to $\mathbf{v}_1$ it is repelled. If the initial point is closer to $\mathbf{v}_1$, it

Dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} 1.25 & -.75 \\ -.75 & 1.25 \end{bmatrix}$$

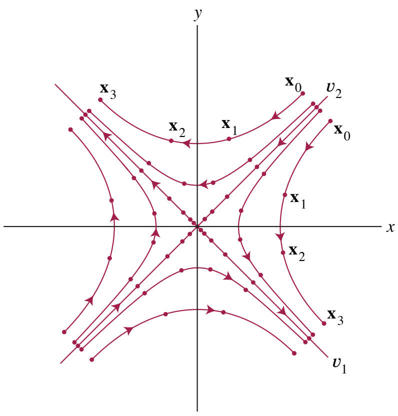


FIGURE 4 The origin as a saddle point.

Example 3

Describe the trajectories of the dynamical system associated to the matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

The characteristic polynomial for A is $(4-\lambda)^2 - 1 = \lambda^2 - 8\lambda + 15 = (\lambda - 5)(\lambda - 3)$. Thus the eigenvalues are 5 and 3 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Hence for any initial vector

$$\mathbf{x}_0 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

we have

$$\mathbf{x}_k = c_1 5^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 3^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

As k becomes large, so do both 5^k and 3^k. Hence $\mathbf{x}_k$ tends away from the origin.

Because the dominant eigenvalue 5 has corresponding eigenvector $\begin{bmatrix}1\\1\end{bmatrix}$, all trajectories for which $c_1 \neq 0$ will end up in the first or third quadrant.

Trajectories for which $c_2 = 0$ start and stay on the line $y = x$ whose direction vector is $\begin{bmatrix}1\\1\end{bmatrix}$. (They move away from $\mathbf{0}$ along this line, unless $\mathbf{x}_0 = \mathbf{0}$).

Similarly, trajectories for which $c_1 = 0$ start and stay on the line $y = -x$ whose direction vector is $\begin{bmatrix}-1\\1\end{bmatrix}$.

In this case $\mathbf{0}$ is called a repellor. This occurs whenever all eigenvalues have modulus greater than 1.

Dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix}1.44 & 0\\0 & 1.2\end{bmatrix}$$

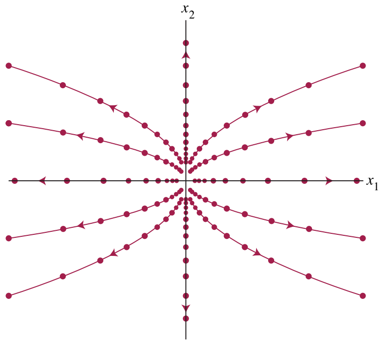


FIGURE 2 The origin as a repellor.

Example 4

Describe the trajectories of the dynamical system associated to the matrix $A = \begin{bmatrix}0.5 & 0.4\\-0.125 & 1.1\end{bmatrix}$. (This was the final matrix in the owl/rat examples earlier.)

Here the eigenvalues 1 and 0.6 have associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix}4\\5\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}4\\1\end{bmatrix}$. So we have

$$\mathbf{x}_k = c_1\mathbf{v}_1 + 0.6^k c_2\mathbf{v}_2\,.$$

As $k \to \infty$, we have $\mathbf{x}_k$ approaching the fixed point $c_1\mathbf{v}_1$.

This situation is unstable – a small change to the entries can have a major effect on the behaviour.

For example with $A := \begin{bmatrix} 0.5 & 0.4 \\ (-0.125) & 1.1 \end{bmatrix}$

value	*eigenvalue*	*eigenvalue*	*behaviour*
-0.125	1	0.6	$\mathbf{x}_k \to c_1\mathbf{v}_1$
-0.1249	1.0099	0.5990	*saddle point*
-0.1251	0.9899	0.6010	$\mathbf{x}_k \to 0$

This example comes from a model of populations of a species of owl and its prey (Lay 5.6.4). In spite of the model being very simplistic, the ecological implications of instability are clear.

Complex eigenvalues

What about trajectories in the complex situation?
Consider the matrices

$$(a) \quad A = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad \text{eigenvalues } \lambda = \tfrac{1}{2} + i\tfrac{1}{2},\ \overline{\lambda} = \tfrac{1}{2} - i\tfrac{1}{2}$$

where $|\lambda| = |\overline{\lambda}| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} < 1.$

$$(b) \quad A = \begin{bmatrix} 0.2 & -1.2 \\ 0.6 & 1.4 \end{bmatrix}, \quad \text{eigenvalues } \lambda = \tfrac{4}{5} + i\tfrac{3}{5},\ \overline{\lambda} = \tfrac{4}{5} - i\tfrac{3}{5}$$

where $|\lambda| = |\overline{\lambda}| = \sqrt{(\frac{4}{5})^2 + (\frac{3}{5})^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{1} = 1.$

If we plot the trajectories beginning with $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ for the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, we get some interesting results.

In case (a) the trajectory spirals into the origin, whereas for (b) it appears to follow an elliptical orbit.

For matrices with complex eigenvalues we can summarise as follows:
if A is a real 2×2 matrix with complex eigenvalues $\lambda = a \pm bi$ then the trajectories of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$

- spiral inward if $|\lambda| < 1$ ($\mathbf{0}$ is a spiral attractor),
- spiral outward if $|\lambda| > 1$ ($\mathbf{0}$ is a spiral repellor),
- and lie on a closed orbit if $|\lambda| = 1$ ($\mathbf{0}$ is a orbital centre).

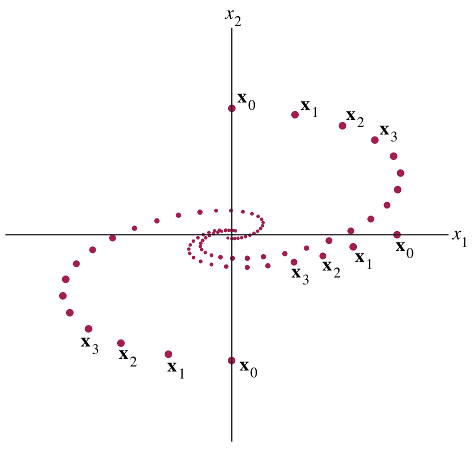


FIGURE 5 Rotation associated with complex eigenvalues.

Some further examples

Example 5

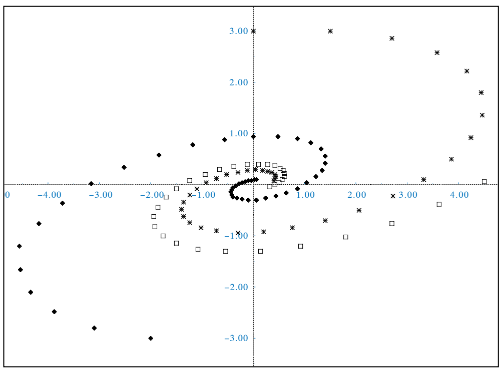
Let $A = \begin{bmatrix} 0.8 & 0.5 \\ -0.1 & 1.0 \end{bmatrix}$.

Here the eigenvalues are $0.9 \pm 0.2i$, with eigenvectors $\begin{bmatrix} 1 \mp 2i \\ 1 \end{bmatrix}$. As we noted in Section 18, setting $P = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, $\cos\varphi = \dfrac{0.9}{\sqrt{0.85}}$, $\sin\varphi = \dfrac{0.2}{\sqrt{0.85}}$,

$$P^{-1}AP = \begin{bmatrix} 0.9 & -0.2 \\ 0.2 & 0.9 \end{bmatrix} = \sqrt{0.85}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

a scaling (approximately 0.92) and a rotation (through approximately 44°).

$P^{-1}AP$ is the matrix of T_A with respect to the basis of the columns of P.
Note that the rotation is anticlockwise.
Here are the trajectories with respect to the original axes. They go clockwise, indicated by $\det(P) < 0$.



Example 6

(Lay 5.6.18) In a herd of buffalo, there are adults, yearlings and calves. On average 42 female calves are borne to every 100 adult females each year, 60% of the female calves survive to become yearlings, and 75% of the female yearlings survive to become adults, and 95% of the adults survive to the next year.

This information gives the following relation:

$$\begin{bmatrix} adults \\ year..s \\ calves \end{bmatrix}_{k+1} = \begin{bmatrix} 0.95 & 0.75 & 0 \\ 0 & 0 & 0.60 \\ 0.42 & 0 & 0 \end{bmatrix} \begin{bmatrix} adults \\ year..s \\ calves \end{bmatrix}_k$$

Assuming that there are sufficient adult males, what are the long term prospects for the herd?

Eigenvalues are approximately

$$1.1048, -0.0774 \pm 0.4063i.$$

The complex eigenvalues have modulus approximately 0.4136.

Corresponding eigenvectors are approximately $\mathbf{v}_1 = \begin{bmatrix} 100.0 \\ 20.65 \\ 38.0 \end{bmatrix}$, and a complex conjugate pair $\mathbf{v}_2, \mathbf{v}_3$.

Thus in the complex setting

$$\begin{aligned} \mathbf{x}_k = \quad 1.1048^k c_1 \mathbf{v}_1 \quad & +(-0.0774 + 0.4063i)^k c_2 \mathbf{v}_2 \\ & +(-0.0774 - 0.4063i)^k c_3 \mathbf{v}_3. \end{aligned}$$

The last two terms go to $\mathbf{0}$ as $k \to \infty$, so in the long term the population of females is determined by the first term, which grows at about 10.5% a year. The distribution of females is 100 adults to 21 yearlings to 38 calves. □

Survival of the Spotted Owls

In the introduction to this chapter the survival of the spotted owl population is modelled by the system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ where

$$\mathbf{x}_k = \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix}$$

where $\mathbf{x}_k$ lists the numbers of females at time k in the juvenile, subadult and adult life stages.
Computations give that the eigenvalues of A are approximately $\lambda_1 = 0.98, \lambda_2 = -0.02 + 0.21i$, and $\lambda_3 = -0.02 - 0.21i$. All eigenvalues are less than 1 in magnitude, since $|\lambda_2|^2 = |\lambda_3|^2 = (-0.02)^2 + (0.21)^2 = 0.0445$.

Denote corresponding eigenvectors by $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. the general solution of $\mathbf{x}_{k+1} = A\mathbf{x}_k$ has the form

$$\mathbf{x}_k = c_1(\lambda_1)^k\mathbf{v}_1 + c_2(\lambda_2)^k\mathbf{v}_2 + c_3(\lambda_3)^k\mathbf{v}_3.$$

Since all three eigenvalues have magnitude less than 1, all the terms on the right of this equation approach the zero vector. So the sequence $\mathbf{x}_k$ also approaches the zero vector.
So this model predicts that the spotted owls will eventually perish.

However if the matrix describing the system looked like

$$\begin{bmatrix} 0 & 0 & 0.33 \\ 0.3 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \quad \text{instead of} \quad \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix}$$

then the model would predict a slow growth in the owl population. The real eigenvalue in this case is $\lambda_1 = 1.01$, with $|\lambda_1| > 1$.
The higher survival rate of the juvenile owls may happen in different areas from the one in which the original model was observed.

Overview

Last time we studied the evolution of a discrete linear dynamical system, and today we begin the final topic of the course (loosely speaking).

Today we'll recall the definition and properties of the dot product. In the next two weeks we'll try to answer the following questions:

Question

What is the relationship between diagonalisable matrices and vector projection? How can we use this to study linear systems without exact solutions?

From Lay, §6.1, 6.2

Motivation for the inner product

- A linear system $A\mathbf{x} = \mathbf{b}$ that arises from experimental data often has no solution. Sometimes an acceptable substitute for a solution is a vector $\hat{\mathbf{x}}$ that makes the distance between $A\hat{\mathbf{x}}$ and $\mathbf{b}$ as small as possible (you can see this $\hat{\mathbf{x}}$ as a good approximation of an actual solution). As the definition for distance involves a sum of squares, the desired $\hat{\mathbf{x}}$ is called a *least squares solution*.
- Just as the dot product on $\mathbb{R}^n$ helps us understand the geometry of Euclidean space with tools to detect angles and distances, the inner product can be used to understand the geometry of abstract vector spaces.

In this section we begin the development of the concepts of orthogonality and orthogonal projections; these will play an important role in finding $\hat{\mathbf{x}}$.

Recall the definition of the dot product:

Definition

The dot (or scalar or inner) product of two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ *in* $\mathbb{R}^n$ *is the scalar*

$$\begin{aligned}(\mathbf{u}, \mathbf{v}) &= \mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} \\ &= \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + \cdots + u_nv_n .\end{aligned}$$

The following properties are immediate:

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

(c) $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(k\mathbf{v})$, $k \in \mathbb{R}$

(d) $\mathbf{u}\cdot\mathbf{u} \geq 0$, $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Example 1

Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$$

Then

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \mathbf{u}^T\mathbf{v} \\ &= \begin{bmatrix} 1 & 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} \\ &= (1)(-1) + (3)(0) + (-2)(3) + (4)(-2) \\ &= -15 \end{aligned}$$

The length of a vector

For vectors in $\mathbb{R}^3$, the dot product recovers the length of the vector:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

We can use the dot product to define the length of a vector in an arbitrary Euclidean space.

Definition

For $\mathbf{u} \in \mathbb{R}^n$, the *length* of $\mathbf{u}$ is

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{u_1^2 + \cdots + u_n^2}.$$

It follows that for any scalar c, the length of $c\mathbf{v}$ is $|c|$ times the length of $\mathbf{v}$:

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|.$$

Unit Vectors

A vector whose length is 1 is called a **unit vector**
If $\mathbf{v}$ is a non-zero vector, then

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

is a unit vector in the direction of $\mathbf{v}$. To see this, compute

$$\begin{aligned} ||\mathbf{u}||^2 &= \mathbf{u}\cdot\mathbf{u} \\ &= \frac{\mathbf{v}}{||\mathbf{v}||}\cdot\frac{\mathbf{v}}{||\mathbf{v}||} \\ &= \frac{1}{||\mathbf{v}||^2}\mathbf{v}\cdot\mathbf{v} \\ &= \frac{1}{||\mathbf{v}||^2}||\mathbf{v}||^2 \\ &= 1 \end{aligned} \tag{1}$$

Replacing $\mathbf{v}$ by the unit vector $\frac{\mathbf{v}}{||\mathbf{v}||}$ is called *normalising* $\mathbf{v}$.

Example 2

Find the length of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$.

$$||\mathbf{u}|| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{\left(\begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}\right)} = \sqrt{1+9+4} = \sqrt{14}.$$

Orthogonal vectors

The concept of perpendicularity is fundamental to geometry. The dot product generalises the idea of perpendicularity to vectors in $\mathbb{R}^n$.

Definition

The vectors $\mathbf{u}$ *and* $\mathbf{v}$ *are* **orthogonal** *to each other if* $\mathbf{u}\cdot\mathbf{v} = 0$.

Since $\mathbf{0}\cdot\mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathbb{R}^n$, the zero vector is orthogonal to every vector.

Orthogonal complements

Definition

Suppose W is a subspace of $\mathbb{R}^n$. If the vector $\mathbf{z}$ is orthogonal to every $\mathbf{w}$ in W, then $\mathbf{z}$ is *orthogonal to* W.

Example 3

The vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to $W = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

Example 4

We can also see that $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is orthogonal to $\text{Nul}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

Definition

The set of all vectors $\mathbf{x}$ that are orthogonal to W is called the *orthogonal complement* of W and is denoted by $W^\perp$.

$$W^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in W\}$$

From the basic properties of the inner product it follows that

- A vector $\mathbf{x}$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
- $W^\perp$ is a subspace
- $W \cap W^\perp = \mathbf{0}$ since $\mathbf{0}$ is the only vector orthogonal to itself.

Example 5

Let $W = \text{Span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$. Find a basis for $W^\perp$, the orthogonal complement of W.

$W^\perp$ consists of all the vectors $\begin{bmatrix}x\\y\\z\end{bmatrix}$ for which

$$\begin{bmatrix}1\\2\\-1\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = 0.$$

For this we must have $x + 2y - z = 0$, which gives $x = -2y + z$.

Thus

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2y+z\\y\\z\end{bmatrix} = y\begin{bmatrix}-2\\1\\0\end{bmatrix} + z\begin{bmatrix}1\\0\\1\end{bmatrix}.$$

So a basis for $W^\perp$ is given by

$$\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}.$$

Since $W = \text{Span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$, we can check that every vector in $W^\perp$ is orthogonal to every vector in W.

Example 6

Let $V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix} \right\}$. Find a basis for $V^\perp$.

$V^\perp$ consists of all the vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ in $\mathbb{R}^4$ that satisfy the two conditions

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix} = 0$$

This gives a homogeneous system of two equations in four variables:

$$\begin{array}{rrrrl} a & +3b & +3c & +d & = 0 \\ 3a & -b & -c & +3d & = 0 \end{array}$$

Row reducing the augmented matrix we get

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 1 & 0 \\ 3 & -1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

So c and d are free variables and the general solution is

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -d \\ -c \\ c \\ d \end{bmatrix} = d \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

The two vectors in the parametrisation above are linearly independent, so a basis for $V^\perp$ is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Notice that in the previous example (and also in the one before it) we found the orthogonal complement as the null space of a matrix.
We have

$$V^\perp = \text{Nul } A$$

where

$$A = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 3 & -1 & -1 & 3 \end{bmatrix}$$

is the matrix whose ROWS are the transpose of the column vectors in the spanning set for V.
To find a basis for the null space of this matrix we just proceeded as usual by bringing the augmented matrix for $A\mathbf{x} = \mathbf{0}$ to reduced row echelon form.

Theorem

Let A be an $m \times n$ matrix.
The orthogonal complement of the row space of A is the null space of A.
The orthogonal complement of the column space of A is the null space of A^T.

$$(Row\ A)^{\perp} = Nul\ A \quad and \quad (Col\ A)^{\perp} = Nul\ A^T.$$

(Remember, Row A is the span of the rows of A.)
Proof The calculation for computing $A\mathbf{x}$ (multiply each row of A by the column vector $\mathbf{x}$) shows that if $\mathbf{x}$ is in Nul A, then $\mathbf{x}$ is orthogonal to each row of A. Since the rows of A span the row space, $\mathbf{x}$ is orthogonal to every vector in Row$A$.
Conversely, if $\mathbf{x}$ is orthogonal to Row A, then $\mathbf{x}$ is orthogonal to each row of A, and hence $A\mathbf{x} = \mathbf{0}$.
The second statement follows since Row A^T = Col A.

Example 7

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$.

- Then Row A = Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.
- Nul A = Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Hence (Row $A)^{\perp}$ = Nul A.

Recall $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$.

- Col A = Span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.
- Nul A^T = Span $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

Clearly, (Col $A)^{\perp}$ = Nul A^T.

An important consequence of the previous theorem.

Theorem

If W is a subspace of $\mathbb{R}^n$, then $\dim W + \dim W^\perp = n$

Choose vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p$ such that $W = \text{Span}\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$. Let

$$A = \begin{bmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \\ \vdots \\ \mathbf{w}_p^T \end{bmatrix}$$

be the matrix whose rows are $\mathbf{w}_1^T, \ldots, \mathbf{w}_p^T$.
Then $W = \text{Row}\, A$ and $W^\perp = (\text{Row}\, A)^\perp = \text{Nul}\ A$. Thus

$$\dim W = \dim(\text{Row}\, A) = \text{Rank}\, A$$

$$\dim W^\perp = \dim(\text{Nul}\ A)$$

and the Rank Theorem implies

$$\dim W + \dim W^\perp = \text{Rank}\, A + \dim(\text{Nul}\ A) = n$$

Example 8

Let $W = \text{Span}\left\{\begin{bmatrix}1\\4\\3\end{bmatrix}\right\}$. Describe $W^\perp$.

We see first that $\dim W = 1$ and W is a *line* through the origin in $\mathbb{R}^3$.
Since we must have $\dim W + \dim W^\perp = 3$, we can then deduce that $\dim W^\perp = 2$: $W^\perp$ is a *plane* through the origin.
In fact, $W^\perp$ is the set of all solutions to the homogeneous equation coming from this equation:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}1\\4\\3\end{bmatrix} = 0.$$

That is,

$$x + 4y + 3z = 0\,.$$

We recognise this as the equation of the plane through the origin in $\mathbb{R}^3$ with normal vector $\langle 1, 4, 3\rangle = \mathbf{w}$.

Basis Theorem

Theorem

If $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_m}\}$ is a basis for W and $\mathcal{C} = \{\mathbf{c_1}, \ldots, \mathbf{c_r}\}$ is a basis for $W^\perp$, then $\{\mathbf{b_1}, \ldots, \mathbf{b_m}, \mathbf{c_1}, \ldots, \mathbf{c_r}\}$ is a basis for $\mathbb{R}^{m+r}$.

It follows that if W is a subspace of $\mathbb{R}^n$, then for any vector $\mathbf{v}$, we can write

$$\mathbf{v} = \mathbf{w} + \mathbf{u},$$

where $\mathbf{w} \in W$ and $\mathbf{u} \in W^\perp$.

If W is the span of a nonzero vector in $\mathbb{R}^3$, then w is just the vector projection of v onto this spanning vector.

Example 9

Let $W = \text{Span}\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}\right\}$. Decompose $\mathbf{v} = \begin{bmatrix}2\\1\\1\\3\end{bmatrix}$ as a sum of vectors in W and $W^\perp$.

To start, we find a basis for $W^\perp$ and then write $\mathbf{v}$ in terms of the bases for W and $W^\perp$.
We're given a basis for W in the problem, and

$$W^\perp = \text{Span}\left\{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\-1\end{bmatrix}\right\}$$

Therefore $\mathbf{v} = 2\left(\begin{bmatrix}1\\1\\0\\1\end{bmatrix}\right) + \left(\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} - \begin{bmatrix}1\\0\\-1\\-1\end{bmatrix}\right) = \begin{bmatrix}2\\2\\0\\2\end{bmatrix} + \begin{bmatrix}0\\-1\\1\\1\end{bmatrix}$.

Overview

Last time

- we defined the dot product on $\mathbb{R}^n$;
- we recalled that the word "orthogonal" describes a relationship between two vectors in $\mathbb{R}^n$;
- we extended the definition of the word "orthogonal" to describe a relationship between a vector and a subspace;
- we defined the *orthogonal complement* $W^\perp$ of the the subspace W to be the subspace consisting of all the vectors orthogonal to W.

Today we'll extend the definition of the word "orthogonal" yet again. We'll also see how orthogonality can determine a particularly useful basis for a vector space.

From Lay, §6.2

Definition of an orthogonal set

Definition

A set $S \subset \mathbb{R}^n$ is *orthogonal* if its elements are pairwise orthogonal.

Example 1

Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}.$$

To show that U is an orthogonal set we need to show that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0, \mathbf{u}_1 \cdot \mathbf{u}_3 = 0$ and $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$.

Example 2

The set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where

$$\mathbf{w}_1 = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}$$

is not an orthogonal set.

We note that $\mathbf{w}_1 \cdot \mathbf{w}_2 = 0$, $\mathbf{w}_1 \cdot \mathbf{w}_3 = 0$ but $\mathbf{w}_2 \cdot \mathbf{w}_3 = -32 \neq 0$.

Theorem (1)

*If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set of **nonzero** vectors in $\mathbb{R}^n$, then S is a linearly independent set, and hence is a basis for the subspace spanned by S.*

Proof:
Suppose that $c_1, c_2, \ldots, c_k$ are scalars such that

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Then

$$\begin{aligned} 0 &= \mathbf{0}\cdot\mathbf{v_1} = (\mathbf{c_1 v_1} + \cdots + \mathbf{c_k v_k})\cdot\mathbf{v_1} \\ &= c_1(\mathbf{v}_1\cdot\mathbf{v}_1) + c_2(\mathbf{v}_2\cdot\mathbf{v}_1) + \cdots + c_k(\mathbf{v}_k\cdot\mathbf{v}_1) \\ &= c_1(\mathbf{v}_1\cdot\mathbf{v}_1) \end{aligned}$$

since $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2, \ldots, \mathbf{v}_k$.
Since $\mathbf{v}_1$ is nonzero, $\mathbf{v}_1\cdot\mathbf{v}_1$, and so $c_1 = 0$.
A similar argument shows that $c_2, \ldots, c_k$ must be zero.
Thus S is linearly independent. □

Definition

An *orthogonal basis* for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthogonal set.

Example 3

Given $\begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\3\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\-1\end{bmatrix}$, find a nonzero vector $\mathbf{x} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ so that the four vectors form an orthogonal set.

We are looking for a vector that satisfies the three conditions

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\cdot\begin{bmatrix}1\\2\\1\\0\end{bmatrix} = 0, \quad \begin{bmatrix}a\\b\\c\\d\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\1\\3\end{bmatrix} = 0, \quad \begin{bmatrix}a\\b\\c\\d\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\0\\-1\end{bmatrix} = 0$$

This gives a homogeneous system of three equations in the four variables a, b, c, d, which reduces the problem to one we already know how to solve.

We solve the system

$$\begin{array}{rrrrl} a & +2b & +c & & = 0 \\ a & - \ b & +c & +3d & = 0 \\ 2a & - \ b & & - \ d & = 0. \end{array}$$

The coefficient matrix of this system is

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & -1 \end{bmatrix}$$

the matrix whose rows are the transpose of the given vectors and the orthogonality condition is indeed $A\mathbf{x} = \mathbf{0}$ (which gives the above system).

Row reducing the augmented matrix of this system we get

$$[A|\mathbf{0}] = \left[\begin{array}{rrrr|r} 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 3 & 0 \\ 2 & -1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

Thus d is free, and $a = b = d$, $c = -3d$.

So the general solution to the system is $\mathbf{x} = d \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$ and every choice of $d \neq 0$ gives a vector as required. For example taking $d = 1$ we get the orthogonal set

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} \right\}$$

This is an orthogonal basis for $\mathbb{R}^4$.

An advantage of working with an orthogonal basis is that the coordinates of a vector with respect to that basis are easily determined.

Theorem (2)

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$, and let $\mathbf{w}$ be any vector in W. Then the unique scalars $c_1, \ldots, c_k$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

are given by

$$c_i = \frac{\mathbf{w}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i} \quad \text{for } i = 1, \ldots, k.$$

Proof Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for W, we know that there are unique scalars $c_1, c_2, \ldots, c_k$ such that $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$.
To solve for c_1, we take the dot product of this linear combination with $\mathbf{v}_i$:

$$\begin{aligned}\mathbf{w}\cdot\mathbf{v}_1 &= (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k)\cdot\mathbf{v}_1 \\ &= c_1(\mathbf{v}_1\cdot\mathbf{v}_1) + \cdots + c_i(\mathbf{v}_i\cdot\mathbf{v}_1) + \cdots + c_k(\mathbf{v}_k\cdot\mathbf{v}_1) \\ &= c_1(\mathbf{v}_1\cdot\mathbf{v}_1)\end{aligned}$$

since $\mathbf{v}_j\cdot\mathbf{v}_1 = 0$ for $j \neq 1$.
Since $\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{v}_1\cdot\mathbf{v}_1 \neq 0$. Dividing by $\mathbf{v}_1\cdot\mathbf{v}_1$, we obtain the desired result

$$c_1 = \frac{\mathbf{w}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}.$$

Similar results follow for $c = 2, \ldots, k$.

□

Example 4

Consider the orthogonal basis for $\mathbb{R}^3$:

$$\mathcal{U} = \left\{ \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}.$$

Express $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ in $\mathcal{U}$ coordinates.

First, check that $\mathcal{U}$ really is an orthogonal basis for $\mathbb{R}^3$:

$$\mathbf{u}_1\cdot\mathbf{u}_2 = \mathbf{u}_1\cdot\mathbf{u}_3 = \mathbf{u}_2\cdot\mathbf{u}_3 = 0.$$

Hence the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set, and since none of the vectors is the zero vector, the set is linearly independen a basis for $\mathbb{R}^3$.

Recall from Theorem (2) that the $\mathbf{u}_i$ coordinate of $\mathbf{x}$ is given by $\dfrac{\mathbf{x}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}$. We compute

$$\mathbf{x}\cdot\mathbf{u}_1 = 6, \quad \mathbf{x}\cdot\mathbf{u}_2 = 13, \quad \mathbf{x}\cdot\mathbf{u}_3 = 2,$$

$$\mathbf{u}_1\cdot\mathbf{u}_1 = 18, \quad \mathbf{u}_2\cdot\mathbf{u}_2 = 9, \quad \mathbf{u}_3\cdot\mathbf{u}_3 = 18.$$

Hence

$$\begin{aligned}\mathbf{x} &= \frac{\mathbf{x}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{x}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 \\ &= \frac{6}{18}\mathbf{u}_1 + \frac{13}{9}\mathbf{u}_2 + \frac{2}{18}\mathbf{u}_3 \\ &= \frac{1}{3}\mathbf{u}_1 + \frac{13}{9}\mathbf{u}_2 + \frac{1}{9}\mathbf{u}_3.\end{aligned}$$

So $\mathbf{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{13}{9} \\ \frac{1}{9} \end{bmatrix}_{\mathcal{U}}$.

Finally, note that if $P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix}$, then

$$P^T P = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 18 \end{bmatrix}.$$

The diagonal form is because the vectors form an orthogonal set, diagonal entries are the squares of the lengths of the vectors. □

Orthonormal sets

Definition

A set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an *orthonormal set* if it is an orthogonal set of unit vectors.

The simplest example of an orthonormal set is the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^n$.
When the vectors in an orthogonal set of nonzero vectors are *normalised* to have unit length, the new vectors will still be orthogonal, and hence the new set will be an orthonormal set.

Recall that in the last example, when P was a matrix with orthogonal columns, $P^T P$ was diagonal. When the columns of a matrix are vectors in an orthonormal set, the situation is even nicer:

Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set in $\mathbb{R}^3$ and $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$. Then

$$U^T U = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{u}_2 & \mathbf{u}_1^T\mathbf{u}_3 \\ \mathbf{u}_2^T\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{u}_2 & \mathbf{u}_2^T\mathbf{u}_3 \\ \mathbf{u}_3^T\mathbf{u}_1 & \mathbf{u}_3^T\mathbf{u}_2 & \mathbf{u}_3^T\mathbf{u}_3 \end{bmatrix}.$$

Hence

$$U^T U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since U is a square matrix, the relation $U^T U = I$ implies that $U^T = U^{-1}$ and thus we also have $UU^T = I$.

In fact,

A **square** matrix U has orthonormal columns if and only if U is invertible with $U^{-1} = U^T$.

Definition

A **square** matrix U which is invertible and such that $U^{-1} = U^T$ is called an **orthogonal matrix**.

It follows from the result above that an orthogonal matrix is a square matrix whose columns form an **orthonormal** set (not just an orthogonal set as the name might suggest).

More generally, we have the following result:

Theorem (3)

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

We also have the following theorem

Theorem (4)

Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$. Then

(1) $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

(2) $(U\mathbf{x})\cdot(U\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$.

(3) $(U\mathbf{x})\cdot(U\mathbf{y}) = 0$ *if and only if* $\mathbf{x}\cdot\mathbf{y} = 0$.

Properties (1) and (3) say that if U has orthonormal columns then the linear transformation $\mathbf{x} \to U\mathbf{x}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) preserves lengths and orthogonality.

Examples

Example 5

The 4×3 matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}$$

has orthogonal columns and $A^T A$ equals

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 3 \\ 2 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Note that here the rows of A are NOT orthogonal. For example, if we take the dot product of the first two rows we get

$$\langle 1, 1, 2\rangle \cdot \langle 2, -1, -1\rangle = 2 - 1 - 2 = -1 \neq 0\,.$$

Now consider the new matrix where each column of A is normalised:

$$B = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{12} & 2/\sqrt{6} \\ 2/\sqrt{6} & -1/\sqrt{12} & -1/\sqrt{6} \\ 1/\sqrt{6} & 1/\sqrt{12} & 0 \\ 0 & 3/\sqrt{12} & -1/\sqrt{6} \end{bmatrix}.$$

Then

$$B^T B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 6

Determine a, b, c such that

$$\begin{bmatrix} a & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ b & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ c & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

is an orthogonal matrix.
The given 2nd and 3rd columns are orthonormal.

So we need to satisfy:

(1) $a^2 + b^2 + c^2 = 1$,

(2) $a/\sqrt{2} + b/\sqrt{6} + c/\sqrt{3} = 0$ which is equivalent to

$$\sqrt{3}a + b + \sqrt{2}c = 0$$

(3) $-a/\sqrt{2} + b/\sqrt{6} + c/\sqrt{3} = 0$ which is equivalent to

$$-\sqrt{3}a + b + \sqrt{2}c = 0.$$

From (2) and (3) we get $a = 0, b = -\sqrt{2}c$.
Substituting in (1) we get $2c^2 + c^2 = 1$ that is $c^2 = \frac{1}{3}$ which gives $c = \pm\frac{1}{\sqrt{3}}$. Thus possible 1st columns are $\pm \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ (there are only two possibilities). □

Overview

Last time we introduced the notion of an orthonormal basis for a subspace. We also saw that if a square matrix U has orthonormal columns, then U is invertible and $U^{-1} = U^T$. Such a matrix is called an *orthogonal* matrix.

At the beginning of the course we developed a formula for computing the projection of one vector onto another in $\mathbb{R}^2$ or $\mathbb{R}^3$. Today we'll generalise this notion to higher dimensions.

From Lay, §6.3

Review

Recall from Stewart that if $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then

$\text{proj}_{\mathbf{u}}\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$.

(Lay uses the notation " $\hat{\mathbf{y}}$ " for this projection, where $\mathbf{u}$ is understood.)

How would you describe the vector $\text{proj}_{\mathbf{u}}\mathbf{y}$ in words?
One possible answer:

> $\mathbf{y}$ *can be written as the sum of a vector parallel to* $\mathbf{u}$ *and a vector orthogonal to* $\mathbf{u}$*;* $\text{proj}_{\mathbf{u}}\mathbf{y}$ *is the summand parallel to* $\mathbf{u}$*.*

Or alternatively,

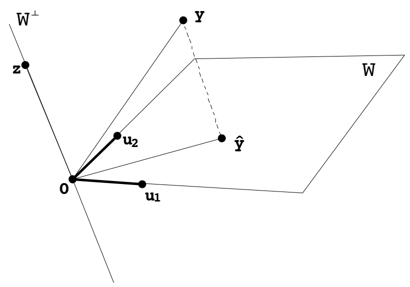
> $\mathbf{y}$ *can be written as the sum of a vector in the line spanned by* $\mathbf{u}$ *and a vector orthogonal to* $\mathbf{u}$*;* $\text{proj}_{\mathbf{u}}\mathbf{y}$ *is the summand in Span*$\{\mathbf{u}\}$*.*

We'd like to generalise this, replacing Span$\{\mathbf{u}\}$ by an arbitrary subspace:

Given $\mathbf{y}$ and a subspace W in $\mathbb{R}^n$, we'd like to write $\mathbf{y}$ as a sum of a vector in W and a vector in $W^{\perp}$.

Example 1

Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ and let $W = \text{Span}\ \{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector $\hat{\mathbf{y}}$ in W and a vector $\mathbf{z}$ in $W^{\perp}$.



Recall that for any orthogonal basis, we have

$$\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3.$$

It follows that

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2$$

and

$$\mathbf{z} = \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3.$$

Since $\mathbf{u}_3$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$, its scalar multiples are orthogonal to $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Therefore $\mathbf{z} \in W^\perp$

All this can be generalised to any vector $\mathbf{y}$ and subspace W of $\mathbb{R}^n$, as we will see next.

The Orthogonal Decomposition Theorem

Theorem

Let W be a subspace in $\mathbb{R}^n$. Then each $\mathbf{y} \in \mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \qquad (1)$$

where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^\perp$.

If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is any orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \qquad (2)$$

*The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of $\mathbf{y}$ onto** W.*

Note that it follows from this theorem that to calculate the decomposition $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, it is enough to know one orthogonal basis for W explicitly. Any orthogonal basis will do, and all orthogonal bases will give the same decomposition $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$.

Example 2

Given

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

let W be the subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Write $\mathbf{y} = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}$ as the sum of a vector in W and a vector orthogonal to W.

The orthogonal projection of $\mathbf{y}$ onto W is given by

$$\begin{aligned}\hat{\mathbf{y}} &= \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 \\ &= \frac{-2}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + \frac{7}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + \frac{6}{3}\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix}5\\-8\\13\\3\end{bmatrix}\end{aligned}$$

Also

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix}2\\-3\\4\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}5\\-8\\13\\3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\-1\\-1\\0\end{bmatrix}$$

Thus the desired decomposition of $\mathbf{y}$ is

$$\begin{aligned}\mathbf{y} &= \hat{\mathbf{y}} + \mathbf{z} \\ \begin{bmatrix}2\\-3\\4\\1\end{bmatrix} &= \frac{1}{3}\begin{bmatrix}5\\-8\\13\\3\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\-1\\-1\\0\end{bmatrix}.\end{aligned}$$

The Orthogonal Decomposition Theorem ensures that $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$ is in $W^{\perp}$. However, verifying this is a good check against computational mistakes.

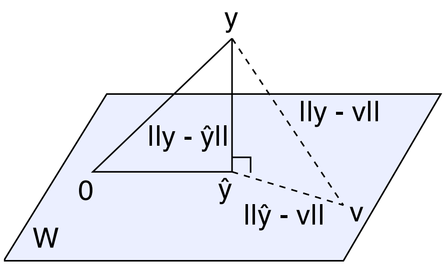
This problem was made easier by the fact that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for W. If you were given an arbitrary basis for W instead of an orthogonal basis, what would you do?

Theorem (The Best Approximation Theorem)

Let W be a subspace of $\mathbb{R}^n$, $\mathbf{y}$ any vector in $\mathbb{R}^n$, and $\hat{\mathbf{y}}$ the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest vector in W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \tag{3}$$

for all $\mathbf{v}$ in W, $\mathbf{v} \neq \hat{\mathbf{y}}$.



Proof

Let $\mathbf{v}$ be any vector in W, $\mathbf{v} \neq \hat{\mathbf{y}}$. Then $\hat{\mathbf{y}} - \mathbf{v} \in W$. By the Orthogonal Decomposition Theorem, $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to W. In particular $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to $\hat{\mathbf{y}} - \mathbf{v}$. Since

$$\mathbf{y} - \mathbf{v} = (\mathbf{y} - \hat{\mathbf{y}}) + (\hat{\mathbf{y}} - \mathbf{v})$$

the Pythagorean Theorem gives

$$\|\mathbf{y} - \mathbf{v}\|^2 = \|\mathbf{y} - \hat{\mathbf{y}}\|^2 + \|\hat{\mathbf{y}} - \mathbf{v}\|^2.$$

Hence $\|\mathbf{y} - \mathbf{v}\|^2 > \|\mathbf{y} - \hat{\mathbf{y}}\|^2$. □

We can now define the distance from a vector $\mathbf{y}$ to a subspace W of $\mathbb{R}^n$.

Definition

Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{y}$ be a vector in $\mathbb{R}^n$. The *distance* from $\mathbf{y}$ to W is

$$||\mathbf{y} - \hat{\mathbf{y}}||$$

where $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto W.

Example 3

Consider the vectors

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

Find the closest vector to $\mathbf{y}$ in $W = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\begin{aligned} \hat{\mathbf{y}} &= \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 \\ &= \frac{30}{10}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \frac{26}{26}\begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}. \end{aligned}$$

Therefore the distance from $\mathbf{y}$ to W is $\left|\left| \begin{bmatrix} 3 \\ -1 \\ \end{bmatrix} - \begin{bmatrix} -1 \\ -5 \\ \end{bmatrix} \right|\right| = \left|\left| \begin{bmatrix} 4 \\ 4 \\ \end{bmatrix} \right|\right| = 8$

Theorem

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ *is an orthonormal basis for a subspace* W *of* $\mathbb{R}^n$*, then for all* $\mathbf{y}$ *in* $\mathbb{R}^n$ *we have*

$$proj_W \mathbf{y} = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p.$$

This theorem is an easy consequence of the usual projection formula:

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p.$$

When each $\mathbf{u}_i$ is a unit vector, the denominators are all equal to 1.

Theorem

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ *is an orthonormal basis for* W *and* $U = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_p\end{bmatrix}$*, then for all* $\mathbf{y}$ *in* $\mathbb{R}^n$ *we have*

$$proj_W \mathbf{y} = UU^T\mathbf{y}\,. \tag{4}$$

The proof is a matrix calculation; see the posted slides for details.

Note that if U is a $n \times p$ matrix with orthonormal columns, then we have $U^TU = I_p$ (see Lay, Theorem 6 in Chapter 6). Thus we have

$$U^TU\mathbf{x} = I_p\mathbf{x} = \mathbf{x} \qquad \text{for every } \mathbf{x} \text{ in } \mathbb{R}^p$$

$$UU^T\mathbf{y} = \text{proj}_W\mathbf{y} \qquad \text{for every } \mathbf{y} \text{ in } \mathbb{R}^n, \text{ where } W = \text{Col } U.$$

Note: Pay attention to the sizes of the matrices involved here. Since U is $n \times p$ we have that U^T is $p \times n$. Thus U^TU is a $p \times p$ matrix, while UU^T is an $n \times n$ matrix.

The previous theorem shows that the function which sends $\mathbf{x}$ to its orthogonal projection onto W is a linear transformation. The kernel of this transformation is ...

...the set of all vectors orthogonal to W, i.e., $W^\perp$.

The range is W itself.

The theorem also gives us a convenient way to find the closest vector to $\mathbf{x}$ in W: find an orthonormal basis for W and let U be the matrix whose columns are these basis vectors. Then mutitply $\mathbf{x}$ by UU^T.

Examples

Example 4

Let $W = \text{Span}\left\{\begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\2\\1\end{bmatrix}\right\}$ and let $\mathbf{x} = \begin{bmatrix}4\\8\\1\end{bmatrix}$. What is the closest vector to $\mathbf{x}$ in W?

Set $\mathbf{u}_1 = \begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$,

$$U = \begin{bmatrix}2/3 & -2/3\\1/3 & 2/3\\2/3 & 1/3\end{bmatrix}.$$

We check that $U^TU = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, so U has orthonormal columns.

The closest vector is

$$\text{proj}_W\mathbf{x} = UU^T\mathbf{x} = \frac{1}{9}\begin{bmatrix}8 & -2 & 2\\-2 & 5 & 4\\2 & 4 & 5\end{bmatrix}\begin{bmatrix}4\\8\\1\end{bmatrix} = \begin{bmatrix}2\\4\\5\end{bmatrix}.$$

We can also compute distance from $\mathbf{x}$ to W:

$$\|\mathbf{x} - \text{proj}_W\mathbf{x}\| = \| \begin{bmatrix}4\\8\\1\end{bmatrix} - \begin{bmatrix}2\\4\\5\end{bmatrix} \| = \| \begin{bmatrix}2\\4\\-4\end{bmatrix} \| = 6.$$

Because this example is about vectors in $\mathbb{R}^3$, so we could also use cross products:

$$\begin{bmatrix}2\\1\\2\end{bmatrix} \times \begin{bmatrix}-2\\2\\1\end{bmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & 1 & 2\\-2 & 2 & 1\end{vmatrix} = -3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k} = \mathbf{n}$$

gives a vector orthogonal to W, so the distance is the length of the projection of $\mathbf{x}$ onto $\mathbf{n}$:

$$\begin{bmatrix}4\\8\\1\end{bmatrix} \cdot \begin{bmatrix}-1/3\\-2/3\\2/3\end{bmatrix} = -6,$$

and the closest vector is

$$\begin{bmatrix}4\\8\\1\end{bmatrix} + 6\begin{bmatrix}-1/3\\-2/3\\2/3\end{bmatrix} = \begin{bmatrix}2\\4\\5\end{bmatrix}.$$

This example showed that the standard matrix for projection to $W = \text{Span}\left\{\begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\2\\1\end{bmatrix}\right\}$ is $\frac{1}{9}\begin{bmatrix}8 & -2 & 2\\-2 & 5 & 4\\2 & 4 & 5\end{bmatrix}$.

If we instead work with $\mathcal{B} = \left\{\begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-2\\2\end{bmatrix}\right\}$ coordinates, what is the orthogonal projection matrix?

Observe that the three basis vectors were chosen very carefully: $\mathbf{b}_1$ and $\mathbf{b}_2$ span W, and $\mathbf{b}_3$ is orthogonal to W. Thus each of the basis vectors is an eigenvector for the linear transformation. (Why?)

The linear transformation is represented by a diagonal matrix when it's written in terms of an eigenbasis. Thus we get the matrix $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$.

What does this tell you about orthogonal projection matrices in general?

Example 5

$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$ are orthogonal and span a subspace W of $\mathbb{R}^4$. Find a vector orthogonal to W.

Normalize the columns and set

$$U = \begin{bmatrix}1/\sqrt{2} & 1/2\\0 & 1/2\\1/\sqrt{2} & -1/2\\0 & -1/2\end{bmatrix}.$$

Then the standard matrix for the orthogonal projection is has matrix

$$UU^T = \frac{1}{4}\begin{bmatrix}3 & 1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & 3 & 1\\-1 & -1 & 1 & 1\end{bmatrix}.$$

Thus, choosing a vector $\mathbf{v} = \begin{bmatrix}3\\2\\0\\1\end{bmatrix}$ not in W, the closest vector to $\mathbf{v}$ in W is given by

$$UU^T\begin{bmatrix}3\\2\\0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}5\\2\\1\\-2\end{bmatrix}.$$

In particular, $\mathbf{v} - UU^T\mathbf{v} = \begin{bmatrix}3\\2\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}5\\2\\1\\-2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\2\\-1\\4\end{bmatrix}$ lies in $W^\perp$.

Thus $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\\4\end{bmatrix}$ are orthogonal in $\mathbb{R}^4$, and span a subspace W_1 of dimension 3.

But now we can repeat the process with W_1! This time take

$$U = \begin{bmatrix}1/\sqrt{2} & 1/2 & 1/\sqrt{22}\\ 0 & 1/2 & 2/\sqrt{22}\\ 1/\sqrt{2} & -1/2 & -1/\sqrt{22}\\ 0 & -1/2 & 4/\sqrt{22}\end{bmatrix},$$

$$UU^T = \frac{1}{44}\begin{bmatrix}35 & 15 & 9 & -3\\ 15 & 19 & -15 & 5\\ 9 & -15 & 35 & 3\\ -3 & 5 & 3 & 43\end{bmatrix}.$$

Taking $\mathbf{x} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, $(I_4 - UU^T)\mathbf{x} = 1/44\begin{bmatrix}3\\-5\\-3\\1\end{bmatrix}$ and then

$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\\4\end{bmatrix}, \begin{bmatrix}3\\-5\\-3\\1\end{bmatrix}$ is an orthogonal basis for $\mathbb{R}^4$. □

Overview

Last time we discussed orthogonal projection. We'll review this today before discussing the question of how to find an orthonormal basis for a given subspace.

From Lay, §6.4

Orthogonal projection

Given a subspace W of $\mathbb{R}^n$, you can write any vector $\mathbf{y} \in \mathbb{R}^n$ as

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = \text{proj}_W \mathbf{y} + \text{proj}_{W^\perp} \mathbf{y},$$

where $\hat{\mathbf{y}} \in W$ is the closest vector in W to $\mathbf{y}$ and $\mathbf{z} \in W^\perp$. We call $\hat{\mathbf{y}}$ the *orthogonal projection of* $\mathbf{y}$ *onto* W.

Given an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ for W, we have a formula to compute $\hat{\mathbf{y}}$:

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p.$$

If we also had an orthogonal basis $\{\mathbf{u}_{p+1}, \ldots, \mathbf{u}_n\}$ for $W^\perp$, we could find $\mathbf{z}$ by projecting $\mathbf{y}$ onto $W^\perp$:

$$\mathbf{z} = \frac{\mathbf{y}\cdot\mathbf{u}_{p+1}}{\mathbf{u}_{p+1}\cdot\mathbf{u}_{p+1}}\mathbf{u}_{p+1} + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_n}{\mathbf{u}_n\cdot\mathbf{u}_n}\mathbf{u}_n.$$

However, once we subtract off the projection of $\mathbf{y}$ to W, we're left with $\mathbf{z} \in W^\perp$. We'll make heavy use of this observation today.

Orthonormal bases

In the case where we have an orthonormal basis $\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ for W, the computations are made even simpler:

$$\hat{\mathbf{y}} = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p.$$

If $\mathcal{U} = \{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ is an orthonormal basis for W and U is the matrix whose columns are the $\mathbf{u_i}$, then

- $UU^T\mathbf{y} = \hat{\mathbf{y}}$
- $U^TU = I_p$

The Gram Schmidt Process

The aim of this section is to find an orthogonal basis $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ for a subspace W when we start with a basis $\{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ that is not orthogonal.

Start with $\mathbf{v_1} = \mathbf{x_1}$.
Now consider $\mathbf{x_2}$. If $\mathbf{v_1}$ and $\mathbf{x_2}$ are not orthogonal, we'll modify $\mathbf{x_2}$ so that we get an orthogonal pair $\mathbf{v_1}$, $\mathbf{v_2}$ satisfying

$$\text{Span}\{\mathbf{x_1}, \mathbf{x_2}\} = \text{Span}\{\mathbf{v_1}, \mathbf{v_2}\}.$$

Then we modify $\mathbf{x_3}$ so get $\mathbf{v_3}$ satisfying $\mathbf{v_1} \cdot \mathbf{v_3} = \mathbf{v_2} \cdot \mathbf{v_3} = 0$ and

$$\text{Span}\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\} = \text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}.$$

We continue this process until we've built a new orthogonal basis for W.

Example 1

Suppose that $W = \text{Span}\ \{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$. Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for W.

To start the process we put $\mathbf{v}_1 = \mathbf{x}_1$.
We then find

$$\hat{\mathbf{y}} = \text{proj}_{\mathbf{v}_1}\mathbf{x}_2 = \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \frac{4}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}.$$

Now we define $\mathbf{v}_2 = \mathbf{x}_2 - \hat{\mathbf{y}}$; this is orthogonal to $\mathbf{x}_1 = \mathbf{v}_1$:

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - \hat{\mathbf{y}} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

So $\mathbf{v}_2$ is the component of $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1$. Note that $\mathbf{v}_2$ is in $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ because it is a linear combination of $\mathbf{v}_1 = \mathbf{x}_1$ and $\mathbf{x}_2$.
So we have that

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$$

is an orthogonal basis for W.

Example 2

Suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for a subspace W of $\mathbb{R}^4$. Describe an orthogonal basis for W.

- As in the previous example, we put

$$\mathbf{v}_1 = \mathbf{x}_1 \quad \text{and} \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1.$$

Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for $W_2 = \text{Span}\,\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$.

- Now $\text{proj}_{W_2}\mathbf{x}_3 = \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$ and

$$\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2}\mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

is the component of $\mathbf{x}_3$ orthogonal to W_2. Furthermore, $\mathbf{v}_3$ is in W because it is a linear combination of vectors in W.

- Thus we obtain that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W.

Theorem (The Gram Schmidt Process)

Given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \ldots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. Also

$$\text{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\,\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \quad \text{for } 1 \le k \le p.$$

Example 3

The vectors

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$$

form a basis for a subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W.

Step 1 Put $\mathbf{v}_1 = \mathbf{x}_1$.

Step 2

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
&= \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} - \frac{(-100)}{50}\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}.
\end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W.

To construct an orthonormal basis for W we normalise the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$:

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{50}}\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{\sqrt{54}}\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Then $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for W.

Example 4

Let $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & 3 \end{bmatrix}$. Use the Gram-Schmidt process to find an orthogonal basis for the column space of A.

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be the three columns of A.

Step 1 Put $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$.

Step 2

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{(-36)}{12}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Step 3

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\ &= \begin{bmatrix} 6 \\ 3 \\ 6 \\ 3 \end{bmatrix} - \frac{12}{12}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{24}{12}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}. \end{aligned}$$

Thus an orthogonal basis for the column space of A is given by

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

Example 5

The matrix A is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Use the Gram-Schmidt process to show that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

is an orthogonal basis for Col A.

Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be the three columns of A.

Step 1 Put $\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Step 2

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}.$$

For convenience we take $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$. (This is optional, but it makes $\mathbf{v}_2$ easier to work with in the following calculation.)

Step 3

$$\mathbf{v}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - 0 - \frac{2}{6}\begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \\ 1 \end{bmatrix}$$

For convenience we take $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$.

QR factorisation of matrices

If an $m \times n$ matrix A has linearly independent columns $\mathbf{x}_1, \ldots, \mathbf{x}_n$, then $A = QR$ for matrices

- Q is an $m \times n$ matrix whose columns are an orthonormal basis for Col(A), and
- R is an $n \times n$ upper triangular invertible matrix.

This factorisation is used in computer algorithms for various computations.

In fact, finding such a Q and R amounts to applying the Gram Schmidt process to the columns of A.

(The proof that such a decomposition exists is given in the text.)

Example 6

Let

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

where the columns of Q are obtained by applying the Gram-Schmidt process to the columns of A and then normalising the columns.
Find R such that $A = QR$.

As we have noted before, $Q^TQ = I$ because the columns of Q are orthonormal. If we believe such an R exists, we have

$$Q^TA = Q^T(QR) = (Q^TQ)R = IR = R.$$

Therefore $R = Q^TA$.

In this case,

$$\begin{aligned} R &= Q^TA \\ &= \begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix} \end{aligned}$$

An easy check shows that

$$QR = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix} \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} = A.$$

Example 7

In Example 4 we found that an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

is given by

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

Normalising the columns gives

$$Q = \begin{bmatrix} -1/\sqrt{12} & 3/\sqrt{12} & 1/\sqrt{30} \\ 3/\sqrt{12} & 1/\sqrt{12} & -2/\sqrt{30} \\ 1/\sqrt{12} & 1/\sqrt{12} & 3/\sqrt{30} \\ 1/\sqrt{12} & -1/\sqrt{12} & 4\sqrt{30} \end{bmatrix}.$$

As in the last example

$$\begin{aligned} R &= Q^T A \\ &= \begin{bmatrix} \sqrt{12} & \sqrt{12} & \sqrt{12} \\ 0 & \sqrt{12} & 2\sqrt{12} \\ 0 & 0 & \sqrt{30} \end{bmatrix}. \end{aligned}$$

It is left as an exercise to check that $QR = A$.

Matrix decompositions

We've seen a variety of matrix decompositions this semester:

- $A = PDP^{-1}$
- $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = S_t R_\theta$
- $A = QR$

In each case, we go to some amount of computation work in order to express the given matrix as a product of terms we understand well. The advantages of this can be either conceptual or computational, depending on the context.

Example 8

An orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Find a QR decomposition of A.

To construct Q we normalise the orthogonal vectors. These become the columns of Q:

$$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{12} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{12} \\ 0 & 2/\sqrt{6} & 1/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{bmatrix}$$

Since $R = Q^T A$, we solve

$$\begin{aligned} R = Q^T A &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} & 0 \\ 1/\sqrt{12} & -1/\sqrt{12} & 1/\sqrt{12} & 3/\sqrt{12} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 3/\sqrt{6} & 2/\sqrt{6} \\ 0 & 0 & 4/\sqrt{12} \end{bmatrix} \end{aligned}$$

Check:

$$\begin{aligned} QR &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{12} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{12} \\ 0 & 2/\sqrt{6} & 1/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 3/\sqrt{6} & 2/\sqrt{6} \\ 0 & 0 & 4/\sqrt{12} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Overview

Last time we introduced the Gram Schmidt process as an algorithm for turning a basis for a subspace into an orthogonal basis for the same subspace. Having an orthogonal basis (or even better, an orthonormal basis!) is helpful for many problems associated to orthogonal projection.

Today we'll discuss the "Least Squares Problem", which asks for the best approximation of a solution to a system of linear equations in the case when an exact solution doesn't exist.

From Lay, §6.5

1. Introduction

Problem: What do we do when the matrix equation $A\mathbf{x} = \mathbf{b}$ has no solution $\mathbf{x}$?
Such inconsistent systems $A\mathbf{x} = \mathbf{b}$ often arise in applications, sometimes with large coefficient matrices.

Answer: Find $\hat{\mathbf{x}}$ such that $A\hat{\mathbf{x}}$ is as close as possible to $\mathbf{b}$.

In this situation $A\hat{\mathbf{x}}$ is an *approximation* to $\mathbf{b}$. The **general least squares problem** is to find an $\hat{\mathbf{x}}$ that makes $\|\mathbf{b} - A\hat{\mathbf{x}}\|$ as small as possible.

Definition

For an $m \times n$ matrix A, a *least squares solution* to $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ such that

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\| \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

The name "least squares" comes from $\| \cdot \|^2$ being the sum of the squares of the coordinates.

It is now natural to ask ourselves two questions:

(1) Do least square solutions always exist?
The answer to this question is YES. We will see that we can use the Orthogonal Decomposition Theorem and the Best Approximation Theorem to show that least square solutions always exist.

(2) How can we find least square solutions?
The Orthogonal Decomposition Theorem —and in particular, the uniqueness of the orthogonal decomposition— gives a method to find all least squares solutions.

Solution of the general least squares problem

Consider an $m \times n$ matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$.

- If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a vector in $\mathbb{R}^n$, then the definition of matrix-vector multiplication implies that

  $$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n\,.$$

  So, the vector $A\mathbf{x}$ is the linear combination of the columns of A with weights given by the entries of $\mathbf{x}$.

- For any vector $\mathbf{x}$ in $\mathbb{R}^n$ that we select, the vector $A\mathbf{x}$ is in Col A. We can solve $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b}$ is in Col A.

- If the system $A\mathbf{x} = \mathbf{b}$ is inconsistent it means that $\mathbf{b}$ is NOT in Col A.

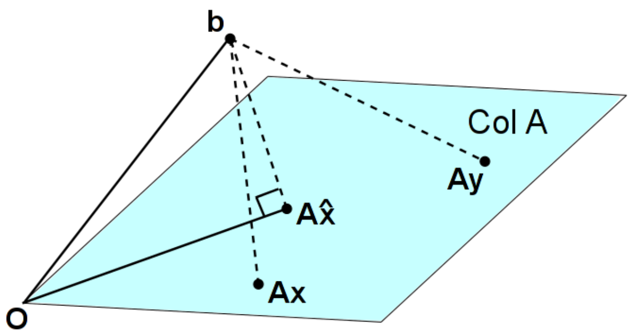
- So we seek $\hat{\mathbf{x}}$ that makes $A\hat{\mathbf{x}}$ the closest point in Col A to $\mathbf{b}$.

- The Best Approximation Theorem tells us that the closest point in Col A to $\mathbf{b}$ is $\hat{\mathbf{b}} = \text{proj}_{\text{Col } A}\mathbf{b}$.

- So we seek $\hat{\mathbf{x}}$ such that $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$. In other words, the least squares solutions of $A\mathbf{x} = \mathbf{b}$ are exactly the solutions of the system

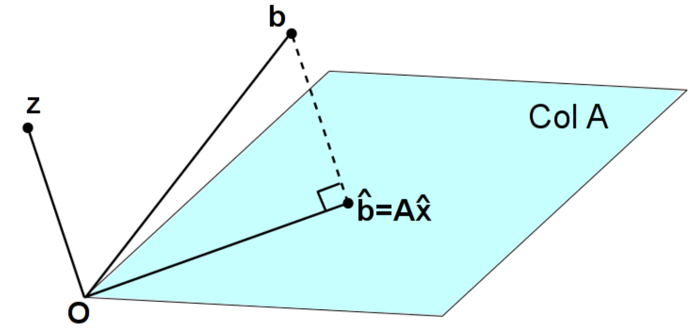
  $$A\hat{\mathbf{x}} = \hat{\mathbf{b}}\,.$$

  By construction, the system $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ is always consistent.



We seek $\hat{\mathbf{x}}$ such that $A\hat{\mathbf{x}}$ is the closest point to $\mathbf{b}$ in Col A.

Equivalently, we need to find $\hat{\mathbf{x}}$ with the property that $A\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{b}$ onto Col(A).



Since $\hat{\mathbf{b}}$ is the closest point to $\mathbf{b}$ in Col A, we need $\hat{\mathbf{x}}$ such that $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$.

The normal equations

- By the Orthogonal Decomposition Theorem, the projection $\hat{\mathbf{b}}$ is the unique vector in Col A with the property that $\mathbf{b} - \hat{\mathbf{b}}$ is orthogonal to Col A.
- Since for every $\hat{\mathbf{x}}$ in $\mathbb{R}^n$ the vector $A\hat{\mathbf{x}}$ is automatically in Col A, requiring that $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ is the same as requiring that $\mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to Col A.
- This is equivalent to requiring that $\mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to each column of A. This means

$$\mathbf{a}_1^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0, \ \mathbf{a}_2^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0, \ \cdots, \ \mathbf{a}_n^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0.$$

- This gives

$$\begin{aligned} \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} (\mathbf{b} - A\hat{\mathbf{x}}) &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ A^T(\mathbf{b} - A\hat{\mathbf{x}}) &= \mathbf{0} \\ A^T\mathbf{b} - A^TA\hat{\mathbf{x}} &= \mathbf{0} \end{aligned}$$

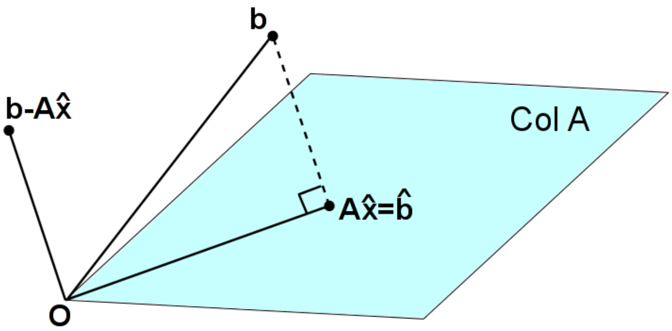
$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$$

These are the normal equations for $\hat{\mathbf{x}}$.

Theorem

The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ coincides with the nonempty set of solutions of the normal equations

$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}.$$



Since $A\hat{\mathbf{x}}$ is automatically in Col A and $\hat{\mathbf{b}}$ is the unique vector in Col A such that $\mathbf{b} - \hat{\mathbf{b}}$ is orthogonal to Col A, requiring that $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ is the same as requiring that $\mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to Col A.

Examples

Example 1

Find a least squares solution to the inconsistent system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}.$$

To solve the normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$, we first compute the relevant matrices:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}.$$

So we need to solve $\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 11 & 14 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 11 & 14 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 8 & 8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right.$$

This gives $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Note that $A\hat{\mathbf{x}} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ and this is the closest point in Col A to $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$.

We could note in this example that $A^TA = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$ is invertible with inverse $\frac{1}{24}\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$. In this case the normal equations give

$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b} \iff \hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}.$$

So we can calculate

$$\begin{aligned} \hat{\mathbf{x}} &= (A^TA)^{-1}A^T\mathbf{b} \\ &= \frac{1}{24}\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Example 2

Find a least squares solution to the inconsistent system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \\ 2 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}.$$

Notice that

$A^TA = \begin{bmatrix} 3 & 1 & 2 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 1 \\ 1 & 14 \end{bmatrix}$ is invertible. Thus the normal equations become

$$\begin{aligned} A^TA\hat{\mathbf{x}} &= A^T\mathbf{b} \\ \hat{\mathbf{x}} &= (A^TA)^{-1}A^T\mathbf{b} \end{aligned}$$

Furthermore,

$$A^T\mathbf{b} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 19 \\ -4 \end{bmatrix}$$

So in this case

$$\begin{aligned} \hat{\mathbf{x}} &= (A^TA)^{-1}A^T\mathbf{b} \\ &= \begin{bmatrix} 14 & 1 \\ 1 & 14 \end{bmatrix}^{-1}\begin{bmatrix} 19 \\ -4 \end{bmatrix} \\ &= \frac{1}{195}\begin{bmatrix} 14 & -1 \\ -1 & 14 \end{bmatrix}\begin{bmatrix} 19 \\ -4 \end{bmatrix} \\ &= \frac{1}{13}\begin{bmatrix} 18 \\ -5 \end{bmatrix}. \end{aligned}$$

With these values, we have

$$A\widehat{\mathbf{x}} = \frac{1}{13}\begin{bmatrix}59\\28\\21\end{bmatrix} \sim \begin{bmatrix}5.54\\2.15\\1.62\end{bmatrix}$$

which is as close as possible to $\begin{bmatrix}4\\3\\2\end{bmatrix}$. □

Example 3

For $A = \begin{bmatrix}1 & 0 & 2\\2 & 1 & 5\\-1 & 1 & -1\\0 & 1 & 1\end{bmatrix}$, what are the least squares solutions to

$A\mathbf{x} = \mathbf{b} = \begin{bmatrix}1\\-1\\-1\\2\end{bmatrix}$?

$$A^TA = \begin{bmatrix}6 & 1 & 13\\1 & 3 & 5\\13 & 5 & 31\end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

For this example, solving $A^TA\hat{x} = A^T\mathbf{b}$ is equivalent to finding the null space of A^TA

$$\begin{bmatrix}6 & 1 & 13\\1 & 3 & 5\\13 & 5 & 31\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

Here, x_3 is free and $x_2 = -x_3, x_1 = -2x_3$.

So $\mathrm{Nul}\ A^TA = \mathbb{R}\begin{bmatrix}2\\1\\-1\end{bmatrix}$.

Here $A\hat{\mathbf{x}} = \mathbf{0}$ –not a very good approximation!
Remember that we are looking for the vectors that map to the closest point to $\mathbf{b}$ in $\mathrm{Col}\ A$.

The question of a "best approximation" to a solution has been reduced to solving the normal equations.

An immediate consequence is that there is going to be a unique least squares solution if and only if $A^T A$ is invertible (note that $A^T A$ is always a square matrix).

Theorem

The matrix $A^T A$ is invertible if and only if the columns of A are linearly independent. In this case the equation $A\mathbf{x} = \mathbf{b}$ has only one least squares solution $\hat{\mathbf{x}}$, and it is given by

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b} \tag{1}$$

For the proof of this theorem see Lay 6.5 Exercises 19 - 21.

Formula (1) for $\hat{\mathbf{x}}$ is useful mainly for theoretical calculations and for hand calculations when $A^T A$ is a 2×2 invertible matrix.

When a least squares solution $\hat{\mathbf{x}}$ is used to produce $A\hat{\mathbf{x}}$ as an approximation to $\mathbf{b}$, the distance from $\mathbf{b}$ to $A\hat{\mathbf{x}}$ is called the **least squares error** of this approximation.

Example 4

Given $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \\ 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ as in Example 2, we found

$$A\hat{\mathbf{x}} = \frac{1}{13}\begin{bmatrix} 59 \\ 28 \\ 21 \end{bmatrix} \sim \begin{bmatrix} 5.54 \\ 2.15 \\ 1.62 \end{bmatrix}$$

Then the **least squares error** is given by $||\mathbf{b} - A\hat{\mathbf{x}}||$, and since

$$\mathbf{b} - A\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 5.54 \\ 2.15 \\ 1.62 \end{bmatrix} = \begin{bmatrix} -1.54 \\ 0.85 \\ 0.38 \end{bmatrix},$$

we have

$$||\mathbf{b} - A\hat{\mathbf{x}}|| = \sqrt{(-1.54)^2 + .85^2 + .38^2} \approx \sqrt{3.24}.$$

Alternative calculations

Note: we didn't cover the QR decomposition in class; these slides are just provided as a reference for your own interest.
In some cases the normal equations for a least squares problem can be *ill conditioned*; that is, small errors in the calculations of the entries of A^TA can sometimes cause relatively large errors in the solution $\hat{\mathbf{x}}$. If the columns of A are linearly independent, the least squares solution can be computed more reliably through a QR factorisation of A.

Theorem

Given an $m \times n$ matrix A with linearly independent columns, let $A = QR$ be a QR factorisation of A. Then for each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution, given by

$$\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}. \tag{2}$$

Proof. Let $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$. Then

$$A\hat{\mathbf{x}} = QR\hat{\mathbf{x}} = QRR^{-1}Q^T\mathbf{b} = QQ^T\mathbf{b}.$$

The columns of Q form an orthonormal basis for Col A. Hence $QQ^T\mathbf{b}$ is the orthogonal projection of $\hat{\mathbf{b}}$ of $\mathbf{b}$ onto Col A.
Thus $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, which shows that $\hat{\mathbf{x}}$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$.
The uniqueness of $\hat{\mathbf{x}}$ follows from the previous theorem. □

Note that $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$ is equivalent to

$$R\hat{\mathbf{x}} = Q^T\mathbf{b} \tag{3}$$

Because R is upper triangular it is faster to solve (3) by back-substitution or row operations than to compute R^{-1} and use (2).

3.1 Examples

Example 5

We are given

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$

Using this QR factoristaion of A we want to find the least squares solution of $A\mathbf{x} = \mathbf{b}$.

We will use the equation $R\hat{\mathbf{x}} = Q^T\mathbf{b}$ to solve this problem.

We calculate

$$Q^T\mathbf{b} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}$$

The least squares solution $\hat{\mathbf{x}}$ satisfies $R\hat{\mathbf{x}} = Q^T\mathbf{b}$; that is

$$\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}.$$

This is easily solved to give

$$\hat{\mathbf{x}} = \begin{bmatrix} 29/10 \\ 9/10 \end{bmatrix},$$

and

$$A\hat{\mathbf{x}} = \begin{bmatrix} 2 \\ 13/2 \\ 2 \\ 13/2 \end{bmatrix}.$$

Example 6

We want to find the least squares solution for $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Gram-Schmidt on the columns of A yields

$$Q = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}.$$

Now we know that $R = Q^TA$.

Thus

$$R = \begin{bmatrix} \sqrt{6} & \sqrt{6}/2 & 11/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}, Q^T\mathbf{b} = \begin{bmatrix} \sqrt{6}/3 \\ 0 \\ -2/\sqrt{3} \end{bmatrix}.$$

So we need to solve

$$\begin{bmatrix} \sqrt{6} & \sqrt{6}/2 & 11/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix} \widehat{\mathbf{x}} = \begin{bmatrix} \sqrt{6}/3 \\ 0 \\ -2/\sqrt{3} \end{bmatrix}$$

Thus $\widehat{\mathbf{x}} = \begin{bmatrix} 5 \\ -2 \\ -2 \end{bmatrix}$ almost immediately. Then $A\widehat{\mathbf{x}} = \mathbf{b}$, an exact solution this time. □