MATH 3325, 2016 — Assignment 1

Due August 12 at 5pm (hand in to the Math 3325 box in the foyer of the Mathematics Department, or submit by email.)

Acknowledgement: Most of these problems are from Stein and Shakarchi Chapter 4, or slight modifications of those problems.

This assignment is out of 100: 4 questions each worth 20 marks, and 20 marks for your writing quality. Acknowledge any help that you receive, either from a book, another student, or an internet source. Discussion of the problems with other students is allowed, but you must write your solutions yourself. Do not look at anyone else's solutions, and do not show your solutions to another student.

(1) (i) (12 marks) Let $\mathcal{L}(H)$ denote the set of bounded linear transformations from a Hilbert space H to itself. Show that $\mathcal{L}(H)$ is complete with respect to the norm

$$\|T\| = \sup_{\|f\|=1} \|Tf\|;$$

that is, show that every Cauchy sequence of operators converges.

(ii) (8 marks) Consider the operator $T : L^2([0,1]) \to L^2([0,1])$ defined by

$$(Tf)(x) = \int_0^x f(s)\, ds.$$

What is the adjoint of T? (There is an explicit formula in terms of integration).

(2) We say that a sequence (A_n) of BLTs on a Hilbert space H converges strongly to the BLT A if

(a) the operator norms $\|A_n\|$ are uniformly bounded, and

(b) for every $f \in H$, $\|A_n f - Af\| \to 0$.

(i) (8 marks) Give an example of a sequence A_n such that A_n converges strongly to the zero operator, but the operator norms $\|A_n - A\|$ do not converge to zero. (Remark: therefore strong convergence, in spite of the name, is weaker than operator norm convergence!)

(ii) (12 marks) Let $T : H \to H$ be a compact operator, and let (A_n) be a sequence of BLTs on H converging strongly to A. Show that $A_n T$ converges in operator norm to AT. Hint: prove by contradiction; that is, suppose that $\|A_n T - AT\|$ does not converge to zero. Express this condition in terms of a sequence f_n of elements of H with norm 1, and then exploit compactness of T.

Remark: as we shall see later in this course, condition (a) actually follows from condition (b), so (a) is redundant and could be omitted.

(3) Consider the following bounded linear operator T on l^2:

$$T(a_1, a_2, \dots) = \Big(a_1, \frac{a_2}{2^3}, \frac{a_3}{3^3}, \dots, \frac{a_n}{n^3}, \dots\Big).$$

We also define $\mathbf{z} \in l^2$ by

$$\mathbf{z} = \Big(1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots\Big).$$

Now, for each real number λ, define a bounded linear transformation T_λ by

$$T_\lambda(\mathbf{x}) = T(\mathbf{x}) + \lambda(\mathbf{x}, \mathbf{z})\mathbf{z}, \quad \mathbf{x} \in l^2.$$

In particular, $T_0 = T$.

(i) (5 marks) Show that T_λ is a compact, self-adjoint operator for every λ.
Remark: it follows that for every λ, there is an orthonormal basis of l^2 consisting of eigenvectors of T_λ.

(ii) (5 marks) Show that if S is a compact self-adjoint operator on l^2 such that all its eigenvalues are nonnegative, then $(S\mathbf{y}, \mathbf{y}) \geq 0$ for every $\mathbf{y} \in l^2$.

(iii) (5 marks) Using the result of (ii), show that for $\lambda < 0$, T_λ has a negative eigenvalue.

(iv) (5 marks) Show that for $\lambda < 0$, T_λ has precisely one negative eigenvalue.

(4) Let B be the unit ball in $\mathbb{R}^d$, and let T be an integral operator on $L^2(B)$ with kernel $K(x, y)$.

(i) (8 marks) Suppose that

$$\sup_y \int_B |K(x, y)|\, dx \leq A \text{ and } \sup_x \int_B |K(x, y)|\, dy \leq A. \tag{0.1}$$

Show that $\|T\| \leq A$. Hint: use the characterization

$$\|T\| = \sup_{\|f\|, \|g\|=1} |(Tf, g)|$$

and use Cauchy-Schwarz and (0.1) on the resulting double integrals.

(ii) (4 marks) Suppose that $K(x, y) = |x - y|^{-d+\alpha}$ where $x, y \in B$ and $\alpha > 0$. Show that T is a bounded operator on $L^2(B)$.

(iii) (8 marks) Show that under the same assumption as in (ii) that T is compact. Hint: Consider the integral operator T_n with kernel

$$K_n(x, y) = |x - y|^{-d+\alpha} 1_{|x-y|>1/n},$$

where $1_{|x-y|>1/n}$ is the characteristic function of the set $\{|x - y| > 1/n\}$.

Note: even if you cannot prove them, use the result of earlier parts of this question to help with later parts of the question.