

## 4 Review of 'calculus'

• Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuous. The support of  $\phi$  is the closure of the set where  $\phi(x) \neq 0$ . If the support of  $\phi$  is compact then we say that  $\phi$  is compactly supported.

• There exist functions  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\phi(x) = 1$  for  $|x| \leq 1$ ,  $\phi$  is  $C^\infty$ , and  $\phi$  is compactly supported. The set of compactly supported, smooth functions on  $\mathbb{R}^n$  is denoted  $C_c^\infty(\mathbb{R}^n)$ .

•  $L^p$  norms. The  $L^p$  norm,  $p \geq 1$ , of a measurable function  $f$  on a measurable set  $E$  is defined to be

$$\|f\|_{L^p(E)} := \left( \int_E |f(x)|^p dx \right)^{1/p}.$$

It is a norm (homogeneous, nonnegative, obeys triangle inequality) provided we identify functions which differ on a set of measure zero. The normed space of (equivalence classes of) functions with finite  $L^p$  norm is denoted  $L^p(E)$ . A very important property is that  $L^p(E)$  is complete; we will prove this later in the course. We also define  $L^\infty(E)$  to be the set of essentially bounded (equivalence classes of) functions, i.e. those for which

$$\|f\|_{L^\infty(E)} := \sup \{M \mid \text{the set } \{x \mid |f(x)| > M\} \text{ has positive measure.}\}$$

is finite. This is also a complete normed space.

- If  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous and compactly supported, then it is in  $L^p$  for every  $1 \leq p \leq \infty$ .
- Hölder's inequality: if  $p^{-1} + q^{-1} = 1$ ,

$$\left| \int_E f(x)g(x) dx \right| \leq \|f\|_{L^p(E)} \|g\|_{L^q(E)}.$$

To prove Hölder's inequality, we begin with Jensen's inequality (stating that secants of convex functions stay above the function) for the function  $x \mapsto b^x$ , obtaining

$$b \leq \frac{1}{p} + \frac{b^q}{q}.$$

Next, we take advantage of the fact that this inequality holds for all  $b$ , but the different terms scale differently in  $b$ . (You should read Terry Tao's blog post 'Amplification, arbitrage, and the tensor product trick!') In particular, replacing  $b$  with  $a^{1-p}b$  and rearranging we obtain Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

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The first part is Tonelli's, the second part Fubini's.

Often we use these in conjunction. Suppose we are asked to integrate some function  $f$  on  $\mathbb{R}^d$ , but don't even know it is integrable. We first apply Tonelli's theorem to  $|f|$ , justifying the use of multiple integrals. Maybe we can calculate them, or if not, at least estimate them. Thus we can establish that  $f$  is integrable. Finally we apply Fubini's theorem to justify using multiple integrals in the actual calculation.

- Polar coordinates:

Let  $f$  be an real-valued integrable function on  $\mathbb{R}^n$ . Define the  $(n-1)$ -sphere by

$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

Let  $\tilde{f}(r, \omega) = f(r\omega)$ , so  $\tilde{f} : \mathbb{R}_+ \times S^{n-1} \rightarrow \mathbb{R}$ . Also, for a measurable subset  $E$  of  $\mathbb{R}^n$ , and  $r > 0$ , define  $E_r \subset S^{n-1}$  by

$$E_r = \{\omega \in S^{n-1} \mid r\omega \in E\}.$$

Then

$$\int_E f(x) dx = \int_0^\infty \left( \int_{E_r} \tilde{f}(r, \omega) d\omega \right) r^{n-1} dr.$$

• Using polar coordinates we see the following: Let  $B$  be the unit ball in  $\mathbb{R}^n$ . The function  $|x|^{-\alpha}$  is in  $L^1(B)$  iff  $\alpha < n$  and it is in  $L^1(\mathbb{R}^n \setminus B)$  iff  $\alpha > n$ .

- Absolutely continuous functions and the fundamental theorem of calculus.

**Definition 4.2.** A function  $f : [a, b] \rightarrow \mathbb{R}$  is *absolutely continuous* if for any  $\epsilon > 0$  there exists a  $\delta > 0$  so that

$$\sum_{k=1}^N |f(b_k) - f(a_k)| < \epsilon \quad \text{whenever} \quad \sum_{k=1}^N (b_k - a_k) < \delta$$

and the intervals  $(a_k, b_k)$  are disjoint.

**Theorem 4.3** (SS Chapter 3, Theorem 3.8). *An absolutely continuous function is differentiable almost everywhere. Moreover, if its derivative is zero almost everywhere, the function is constant.*

**Theorem 4.4** (SS Chapter 3, Theorem 3.11). *The derivative of an absolutely continuous function  $f$  is integrable, and*

$$F(x) - F(a) = \int_a^x F'(y) dy.$$

*Conversely, if  $f$  is integrable on  $[a, b]$ , then  $F(x) = \int_a^x f(y) dy$  is absolutely continuous and  $F'(x) = f(x)$  almost everywhere.*

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From this, Hölder's inequality follows easily – first prove it for functions with  $\|f\|_p = 1$  and  $\|g\|_q = 1$ .

- Dominated convergence theorem. (SS Chapter 2 Theorem 1.13).

Let  $f_n$  be a sequence of functions in  $L^1(E)$  converging pointwise a.e. to  $f$ . Suppose that  $|f_n(x)| \leq g(x)$  for a fixed  $L^1$  function  $g$ . Then

$$\int_E f_n \rightarrow \int_E f.$$

Sketch: Consider the sets  $E_N$  on which  $|x| \leq N$  and  $|g(x)| \leq N$ . Eventually, every point is in some  $E_N$ , and so by the monotone convergence theorem  $\int_{E_N} g$  becomes arbitrarily small. Estimate  $\int_E |f_n - f|$  as the sum of the integral on one of these sets and the integral on the complement; use the bounded convergence theorem on the first integral and  $|f_n - f| \leq 2g$  on the second.  $\square$

The bounded convergence theorem is now a special case of the dominated convergence theorem, but of course one needs to prove it first!

The bounded convergence theorem follows easily from Egorov's theorem (SS Chapter 1 Theorem 4.4) which says that any pointwise limit of functions actually converges uniformly, off some arbitrarily small open set.

Sketch: [Egorov] Define

$$E_k^n = \{x \in E \mid |f_j(x) - f(x)| < 1/n \text{ for all } j > k\}.$$

Choose  $k_n$  large enough that  $m(E - E_{k_n}^n) < 2^{-n}$ . Let  $\tilde{A}$  be the intersection of some tail of the sets  $\{E_{k_n}^n\}$ , choosing the tail so that  $\tilde{A}$  has almost full measure. Finally let  $A$  be a closed subset of  $\tilde{A}$ , omitting only a small set.  $\square$

- Fubini-Tonelli theorem (in  $\mathbb{R}^n$ ):

**Theorem 4.1.**

(i) Suppose that  $f : \mathbb{R}^{n+m} \rightarrow \mathbb{C}$  is nonnegative and measurable. Then

$$\begin{aligned} \int_{\mathbb{R}^{n+m}} f &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} f(x, y) dy \right) dx \\ &= \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} f(x, y) dx \right) dy. \end{aligned} \quad (4.1)$$

*Note: this is an equality in extended real numbers: the left hand side might be  $+\infty$ , but this happens if and only if the right hand side is also  $+\infty$ .*

(ii) Suppose that  $f \in L^1(\mathbb{R}^{n+m})$ . Then (4.1) holds.

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- Differentiating under the integral sign:

**Proposition 4.5.** Suppose that  $U$  is an open set in  $\mathbb{R}^n$ ,  $E$  is a measurable set in  $\mathbb{R}^k$ ,  $f : U \times E \rightarrow \mathbb{R}$  is a function so that

- $f(x, \cdot) : E \rightarrow \mathbb{R}$  is measurable for each  $x \in U$ .
- $\partial_{x_i} f(x, y)$  exists and is continuous for all  $(x, y)$  and
- (iii) (the crucial condition)

$$|\partial_{x_i} f(x, y)| \leq g(y) \text{ for some } g \in L^1(E).$$

Then

$$\frac{\partial}{\partial x_i} \int_E f(x, y) dy = \int_E \frac{\partial f}{\partial x_i}(x, y) dy.$$

Proof: (sketch) The LHS is, for a fixed  $x$ ,

$$\lim_{h \rightarrow 0} \int_E \frac{f(x + h e_i, y) - f(x, y)}{h} dy.$$

Use (ii) and the mean value theorem to write the integrand as  $\partial_{x_i} f(x + \theta(h) e_i, y)$  for some  $0 \leq \theta(h) \leq h$  and conclude that it is pointwise bounded by  $g(y)$ . Then by the dominated convergence theorem, we can take the pointwise limit inside the integral. This is just  $\partial_{x_i} f(x, y)$  using (ii) again, which gives us the RHS.  $\square$

- Change of variable formula:

**Theorem 4.6.** Let  $R \subset \mathbb{R}^n$  be a rectangle, and  $F : R \rightarrow \mathbb{R}^n$  a  $C^1$  function. Then for every continuous function  $f$  defined on  $F(R)$ , we have the change of variable formula

$$\int_{F(R)} f(y) dy = \int_R (f \circ F)(x) |\det DF(x)| dx. \quad (4.2)$$

We sometimes write this differently: we think of  $F$  as relating two different sets of coordinates, the  $y$  coordinates on  $F(R)$  and the  $x$  coordinates on  $R$ . We sometimes write  $y = y(x)$  instead of  $y = F(x)$ . Also, the Jacobian matrix  $DF$  is sometimes written  $\partial y / \partial x$ . So we have

$$\int_{F(R)} f(y) dy = \int_R f(y(x)) \left| \det \frac{\partial y}{\partial x} \right| dx.$$

- Surface measure. Let  $S$  be a hypersurface given by the graph of a  $C^1$  function:

$$S = \{(x_1, \dots, x_n) \mid x_n = u(x_1, x_2, \dots, x_{n-1})\},$$

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$$u \in C^1(\mathbb{R}^{n-1}).$$

Then, in terms of the coordinates  $(x_1, \dots, x_{n-1})$  on  $S$ , surface measure on  $S$  is defined to be

$$d\sigma = \sqrt{1 + |\nabla u(x')|^2} dx', \quad x' = (x_1, \dots, x_{n-1}). \quad (4.3)$$

**Proposition 4.7.** *The measure  $d\sigma$  on  $S$  is invariant under a Euclidean change of coordinates. That is, suppose that  $(y_1, \dots, y_n)$  are another set of Euclidean coordinates. This means that there is an orthonormal basis  $e'_i$  such that  $(y_1, \dots, y_n)$  represents the point  $\sum_i y_i e'_i$ . If  $S$  can also be written as a graph in the  $y$  coordinates,*

$$S = \{(y_1, \dots, y_n) \mid y_n = v(y_1, y_2, \dots, y_{n-1})\}, \quad v \in C^1,$$

then we have

$$d\sigma = \sqrt{1 + |\nabla v(y')|^2} dy', \quad y' = (y_1, \dots, y_{n-1}).$$

The key to proving this proposition is showing that, if the  $y'$  coordinates on  $S$  are given in terms of  $x'$  by  $y' = F(x')$ , then

$$\det DF(x_0) = \frac{\sqrt{1 + |\nabla u(x'_0)|^2}}{\sqrt{1 + |\nabla v(y'_0)|^2}}, \quad y'_0 = F(x'_0). \quad (4.4)$$

We then use Theorem 4.6.

The identity (4.4) can be proved by considering two Euclidean sets of coordinates  $y = (y_1, \dots, y_n)$  and  $x = (x_1, \dots, x_n)$ . Change to  $\tilde{y} = (y_1, \dots, y_{n-1}, Y_n)$  and  $\tilde{x} = (x_1, \dots, x_{n-1}, X_n)$  where  $Y_n = y_n - v(y')$ ,  $X_n = x_n - u(x')$ . Then show that, on the surface,

$$\det \frac{\partial y'}{\partial x'} = \left( \frac{\partial Y_n}{\partial X_n} \right)^{-1}.$$

This can be computed explicitly, to be equal to

$$\frac{\sqrt{1 + |\nabla u(x')|^2}}{\sqrt{1 + |\nabla v(y')|^2}}.$$

- The result above allows us to define surface measure for any  $C^1$  hypersurface, not just a graph.
- Integration by parts: the following result will be adequate for now; it is possible to weaken the assumptions.

**Proposition 4.8.**

(i) Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with  $C^1$  boundary. Then if  $f, g \in C^1(\overline{\Omega})$ , we have

$$\int_{\Omega} \left( f \frac{\partial g}{\partial x_i} + g \frac{\partial f}{\partial x_i} \right) dx = \int_{\partial\Omega} f g n_i d\sigma$$

where  $n_i = n \cdot e_i$  is the  $i$ th component of the outward pointing normal vector  $n$  and  $\sigma$  is surface measure on  $\partial\Omega$ .

(ii) Assume that  $f, g$  are  $C^1$  functions on  $\mathbb{R}^n$ , such that  $f, \partial_{x_i} f \in L^p(\mathbb{R}^n)$ , while  $g, \partial_{x_i} g \in L^q(\mathbb{R}^n)$ , with  $p^{-1} + q^{-1} = 1$ . Then

$$\int_{\mathbb{R}^n} f \frac{\partial g}{\partial x_i} dx = - \int_{\mathbb{R}^n} g \frac{\partial f}{\partial x_i} dx.$$

Notice that  $dx' = (n \cdot e_n) d\sigma$  in the notation of (4.3), where  $n$  is the upward pointing unit normal to  $S$ .