10 Complete normed spaces of functions

We are now in a position to define standard spaces of functions.

Let $p \in [1, \infty)$ and define $L^p(X, \mathcal{M}, \mu)$ to be the space of complex-valued measurable functions f on X such that $|f|^p$ is integrable, modulo the equivalence relation of being equal a.e. w.r.t. μ. This is a vector space which we endow with the p-norm

$$\|f\|_p = \Big(\int_X |f(x)|^p \, d\mu \Big)^{1/p}.$$

We have to check that this is a norm. It is straightforward to see that it satisfies strict positivity and homogeneity, but the triangle inequality is far from obvious!

Lemma 10.1 (Hölder's inequality)**.** *If $p^{-1} + q^{-1} = 1$, then*

$$\Big| \int_X f(x)g(x) \, d\mu \Big| \leq \|f\|_p \|g\|_q.$$

Proof: To show Hölder, it suffices by homogeneity to do this when $\|f\|_p = \|g\|_q = 1$. Then we use Young's equality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b \geq 0,$$

on the LHS, and integrate to find that the LHS is bounded by 1. □

By the way this is also valid (in fact, almost trivial) when $p = 1$ and $q = \infty$, where the L^∞ norm is defined by

$$\|f\|_\infty = \inf \Big\{ M \mid \mu\{x \mid |f(x)| > M\} \text{ has measure zero} \Big\}.$$

Thus $\|f\|_\infty$ is finite if f can be modified on a set of measure zero so that it is bounded by M, and $\|f\|_\infty$ is the smallest M with this property.

To show that $\|\cdot\|_p$ satisfies the triangle inequality, we write

$$S = \Big(\int_X |f + g|^p d\mu \Big)^{1/p}.$$

Then,

$$S^p \leq \int_X |f||f + g|^{p-1} \, d\mu + \int_X |g||f + g|^{p-1} \, d\mu.$$

Applying Hölder's inequality to the expressions on the right hand side:

$$S^p \leq \|f\|_p \|f + g\|_p^{p-1} + \|g\|_p \|f + g\|_p^{p-1} = (\|f\|_p + \|g\|_p) S^{p-1}$$

and rearranging this gives the triangle inequality.

Now, in a normed vector space the infinite triangle inequality always holds. If we have $x = \sum_{i=1}^{\infty} x_i = \lim_{N\to\infty} \sum_{i=1}^{N} x_i$, then

$$||x|| = \lim_{N\to\infty} \left\| \sum_{i=1}^{N} x_i \right\| \leq \lim_{N\to\infty} \sum_{i=1}^{N} ||x_i|| = \sum_{i=1}^{\infty} ||x_i||$$

where in the first equality we use the fact that the norm is by definition continuous (it's perhaps that 'most continuous' function of all!), and in the second step we use the finite term triangle inequality.

However, in what follows we will need to take some care with pointwise limits. Suppose we merely have a function g which is defined via a pointwise limit:

$$g(x) = \sum_{i=1}^{\infty} g_i(x).$$

Can we use the triangle inequality with L^p norm? Suppose that the functions g_i are all non-negative. We can follow through our proof of the triangle inequality above, with one extra step. Let

$$S = \left(\int_X |g(x)|^p d\mu(x) \right)^{1/p}$$

and so

$$\begin{aligned} S^p &= \int_X |g(x)|^p d\mu(x) \\ &= \int_X |g(x)||g(x)|^{p-1} d\mu(x) \\ &= \int_X \lim_{N\to\infty} \left(\left| \sum_{i=1}^{N} g_i(x) \right| \right) |g(x)|^{p-1} d\mu(x) \\ &\leq \int_X \lim_{N\to\infty} \left(\sum_{i=1}^{N} |g_i(x)| \right) |g(x)|^{p-1} d\mu(x) \end{aligned}$$

which by the monotone convergence theorem becomes

$$= \sum_{i=1}^{\infty} \int_X |g_i(x)||g(x)|^{p-1} d\mu(x)$$

and applying the Hölder inequality in each term gives

$$S^p = \sum_{i=1}^{\infty} ||g_i||_p ||g||_p^{p-1}$$
$$= \sum_{i=1}^{\infty} ||g_i||_p S^{p-1},$$

so

$$||g||_p \le \sum_{i=1}^{\infty} ||g_i||_p.$$

Lemma 10.2. *For any Cauchy sequence $\{f_n\}$ in a metric space, there is a subsequence satisfying*

$$d(f_{n_{k+1}}, f_{n_k}) < \epsilon(k),$$

for any positive function $\epsilon(k)$.

Proof: For each k, there is some N_k so $d(f_a, f_b) < \epsilon(k)$ for all $a, b \ge N_k$. Take n_k to be the maximum of $\{N_0, \ldots, N_k\}$. □

Theorem 10.3. *For $1 \le p \le \infty$, the space $L^p(X, \mu)$ is complete under the norm $\| \cdot \|_p$.*

Proof of Theorem 10.3: We first do $p = 1$. Given a Cauchy sequence f_n in $L^1(X, \mu)$, we show that it has a limit in $L^1(X, \mu)$.

By passing to a subsequence, we may assume that

$$\|f_n - f_{n-1}\|_1 \le 2^{-n}.$$

Then the function $g = |f_1| + \sum_{j=1}^{\infty} |f_{j+1} - f_j|$ is an integrable function. As all the terms are positive, we see that this series must be converging pointwise a.e. (the alternative is that on a set of positive measure the series sums to ∞, but then the function would not be integrable). It follows that the series $f_1(x) + \sum_j (f_{j+1}(x) - f_j(x))$ converges a.e., since it converges absolutely a.e. Let $f(x)$ be the limiting function (defined a.e.). Then, $f_n \to f$ pointwise a.e. and f_n is dominated by g, so $\int |f_n - f| \to 0$ by DCT.

For $1 < p < \infty$, we can use essentially the same argument. Passing to a subsequence, we assume

$$\|f_n - f_{n-1}\|_p \le 2^{-n}.$$

Now let

$$g(x) = |f_1(x)| + \sum_{n=2}^{\infty} |f_n(x) - f_{n-1}(x)|.$$

Then $\|g\|_p \le \|f_1\|_p + \sum \left\| |f_n - f_{n-1}| \right\|_p < \infty$. Therefore, g is finite a.e. We can see that

$$f_n(x) = f_1(x) + \sum_{j=2}^{n} (f_j(x) - f_{j-1}(x)),$$

and since the sum

$$f_1(x) + \sum_{j=2}^{\infty} (f_j(x) - f_{j-1}(x))$$

converges for a.e. x, the sequence $\{f_n(x)\}$ converges for a.e. x. Let $f(x)$ be the limit of the sequence $\{f_n(x)\}$. Then we have $f_n \to f$ a.e. and $|f_n|^p \le g^p$, where g^p is an integrable function. Therefore, $|f|^p \le g^p$, i.e. $f \in L^p$. Also, $|f_n - f| \to 0$ a.e. and

$$|f_n(x) - f(x)|^p \le \Big(|f_n(x) + |f(x)|\Big)^p \le 2^p g^p,$$

which is an integrable function. We conclude that $f_n \to f$ in L^p by the dominated convergence theorem.

The proof for $p = \infty$ is different in character. Assume that the sequence f_n is Cauchy in L^∞. Then given k there exists $N(k)$ such that

$$|f_n(x) - f_m(x)| \le 2^{-k} \text{ a.e. for } n, m \ge N(k).$$

Let C_{kmn}, for $n, m \ge N(k)$, be the set where this estimate fails. Then C_{kmn} has measure zero and thus $C = \cup_{k,m,n} C_{kmn}$ has measure zero. For $x \in C^c$, we have

$$|f_n(x) - f_m(x)| \le 2^{-k} \text{ for } n, m \ge N(k). \tag{10.1}$$

Thus, for $x \in C^c$, the sequence $f_n(x)$ is Cauchy. Let $f(x)$ denote the limit of this sequence, and define f arbitrarily (say, equal to 0) for $x \in C$. Then taking $m \to \infty$ in (10.1),

$$|f_n(x) - f(x)| \le 2^{-k} \text{ for } x \in C^c,\ n \ge N(k).$$

Therefore $\|f\|_\infty \le \|f_{N(k)}\|_\infty + 2^{-k}$, so $f \in L^\infty$, and $f_n \to f$ in the L^∞ norm. □