11 Radon-Nikodym derivatives

11.1 Signed measures

Let $(X, \mathcal{M})$ be a measurable space. A signed measure is a map ν from $\mathcal{M}$ to $(-\infty, \infty]$ with the property that if $E_1, E_2, \ldots$ are disjoint elements of $\mathcal{M}$, then

$$\nu(\cup_j E_j) = \sum_{j=1}^{\infty} \nu(E_j).$$

Notice that this implies that if $\nu(\cup_j E_j) < \infty$, then the sum on the RHS is absolutely convergent, for otherwise it would not be independent of the ordering of the E_j. Sometimes we refer to (unsigned) measures as positive measures to make the distinction clear.

An example of a signed measure is

$$\nu(E) = \int_E f d\mu$$

where $(X, \mathcal{M}, \mu)$ is a measure space and f is a fixed real-valued function such that f_-, the negative part of f, is integrable. (This ensures that ν can never take the value $-\infty$, which is not allowed.) In fact, we shall soon prove that this is the only possibility.

Given a signed measure ν, we define the total variation $|\nu| : \mathcal{M} \to \mathbb{R}$ as follows:

$$|\nu|(E) = \sup \sum_{j=1}^{\infty} |\nu(E_j)|,$$

where we sup over all ways of decomposing E into a countable disjoint union of measurable sets E_j.

Proposition 11.1. *The total variation $|\nu|$ is a positive measure satisfying*

$$\nu \leq |\nu|.$$

Proof: We need to show that

$$|\nu|(E) \leq \sum_{j=1}^{\infty} |\nu|(E_j) \text{ and } |\nu|(E) \geq \sum_{j=1}^{\infty} |\nu|(E_j)$$

whenever E is written as a countable disjoint union of measurable sets E_j.

To prove $\geq$, we choose numbers $\alpha_j < |\nu|(E_j)$. Then, we can find a partition $E_j = \cup_i F_{i,j}$ into measurable sets such that

$$\alpha_j \leq \sum_j |\nu(F_{i,j})|.$$

Then $\cup_{i,j} F_{i,j}$ is a partition of E, so we get

$$\sum_j \alpha_j \leq \sum_{i,j} |\nu(F_{i,j})| \leq |\nu|(E).$$

Taking the sup over all possible α_j proves $\geq$.

To prove $\leq$, we take a partition of E into measurable sets F_k. Then we have

$$\begin{aligned}\sum_k |\nu(F_k)| &= \sum_k \Big| \sum_j \nu(F_k \cap E_j) \Big| \\ &\leq \sum_{k,j} |\nu(F_k \cap E_j)| \leq \sum_j |\nu|(E_j).\end{aligned}$$

Since this is true for each way of partitioning E, we find that

$$|\nu|(E) \leq \sum_j |\nu|(E_j)$$

as required.

The statement $\nu \leq |\nu|$ is obvious. □

We can then write any signed measure as the difference of two positive measures, by writing

$$\nu = \frac{\nu + |\nu|}{2} + \frac{\nu - |\nu|}{2} = \nu_+ + \nu_-.$$

We say that ν is σ-finite if $|\nu|$ is, and then ν_+ and ν_- automatically are as well.

Notice that the finite signed measures on a measurable space $(X, \mathcal{M})$ form a vector space, denoted $M(X)$.

Theorem 11.2. *$M(X)$ is a complete normed space under the norm*

$$\|\nu\|_{M(X)} = |\nu|(X).$$

Proof: It is straightforward to show that $\|\cdot\|_{M(X)}$ is a norm. Suppose that ν_j is a Cauchy sequence in $M(X)$. Then for each $E \in \mathcal{M}$, $|\nu_n(E) - \nu_m(E)| \to 0$ as $m, n \to \infty$, so $\lim_n \nu_n(E)$ exists for each E. Define $\nu(E)$ to be $\lim_n \nu_n(E)$. We need to show that ν is a finite signed measure.

To show countable additivity, suppose that $E = \cup_i E_i$ is a disjoint union of measurable sets.

Lemma 11.3. *For $\epsilon > 0$ there exists an $N(\epsilon)$ so*

$$\sum_{i=1}^{\infty} |\nu_n(E_i) - \nu_m(E_i)| < \epsilon \textit{ for } n, m \geq N(\epsilon).$$

Proof: The idea is to rewrite the sum as

$$\sum_{i=1}^{\infty} |\nu_n(E_i) - \nu_m(E_i)| = \sum_{i=1}^{I} |\nu_n(E_i) - \nu_m(E_i)| + \sum_{i>I} |\nu_n(E_i) - \nu_m(E_i)|$$

for any choice of integer I, and then

$$\leq \sum_{i=1}^{I} |\nu_n(E_i) - \nu_m(E_i)| + \sum_{i>I} |\nu_n(E_i)| + \sum_{i>1} |\nu_m(E_i)|$$

for any integer k, and then

$$\leq \sum_{i=1}^{I} |\nu_n(E_i) - \nu_m(E_i)| + 2 \sum_{i>I} |\nu_k(E_i)| + \Big\| |\nu_n| - |\nu_k| \Big\| + \Big\| |\nu_m| - |\nu_k| \Big\|.$$

(Note we made some fairly loose estimates there!) Nevertheless, this is enough, as we see that we can make each term small.

The last two terms are eventually small, because $\{\nu_n\}_n$ is a Cauchy sequence in $M(X)$. In particular, we can find some $M(\epsilon)$ so for all $a, b \geq M(\epsilon)$, we have $\Big\| |\nu_a| - |\nu_b| \Big\| < \epsilon/5$. We now pick our arbitrary integer $k = M(\epsilon)$. Looking at the second term, we see that it is the sum of absolute values of measures (with respect to the fixed measure ν_k of the tail of some sequence of sets). Thus we can choose our arbitrary integer $I = I(\epsilon, M(\epsilon))$ so that $\sum_{i>I} |\nu_k(E_i)| < \epsilon/5$. Finally, the first term is just a finite sum, so we can find some $N(\epsilon) \geq M(\epsilon)$ so that $n, m \geq N(\epsilon)$ implies that $|\nu_n(E_i) - \nu_m(E_i)| < \epsilon/(5I)$, and so the whole first term is at most $\epsilon/5$. □

Certainly then, for every integer M,

$$\sum_{i=1}^{M} |\nu_n(E_i) - \nu_m(E_i)| < \epsilon \text{ for } n, m \geq N(\epsilon).$$

Taking m to infinity, we find that

$$\sum_{i=1}^{M} |\nu_n(E_i) - \nu(E_i)| \leq \epsilon \text{ for } n \geq N(\epsilon).$$

Since this is true for all M, we get

$$\sum_{i=1}^{\infty} |\nu_n(E_i) - \nu(E_i)| \leq \epsilon \text{ for } n \geq N(\epsilon). \tag{11.1}$$

Now we can compute

$$\begin{aligned}|\nu(E) - \sum_i \nu(E_i)| &= \lim_n |\nu_n(E) - \sum_i \nu(E_i)| \\ &= \lim_n |\sum_i (\nu_n(E_i) - \nu(E_i))| \\ &\leq \limsup_n \sum_i |\nu_n(E_i) - \nu(E_i)| \\ &\leq \epsilon\end{aligned}$$

by (11.1). Since this is true for all ϵ, we see that $\nu(E) = \sum_i \nu(E_i)$, so ν is countably additive, and hence a signed measure. Now (11.1) with E_i a partition of X shows that $\|\nu_n - \nu\|_{M(X)} \to 0$. This shows that ν is a finite measure and that $\nu_n \to \nu$ under the total variation norm, completing the proof. □

11.2 Absolute continuity

Definition 11.4.

1. We say that a signed measure μ is supported on a set A if $\mu(E) = \mu(E \cap A)$ for all $E \in \mathcal{M}$.

2. Two signed measures μ and ν are mutually singular if they are supported on disjoint subsets. This is denoted $\mu \perp \nu$.

3. If ν is a signed measure and μ a positive measure, we say that ν is absolutely continuous w.r.t. μ if

$$\mu(E) = 0 \implies \nu(E) = 0.$$

If $|\nu|$ is a finite measure then this last condition is equivalent to the assertion that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) < \delta \implies |\nu|(E) < \epsilon,$$

while in general this is a strictly stronger assertion.

Example. Lebesgue measure, delta measures, and $E \mapsto \int_E f$ on $\mathbb{R}^n$.

Exercise. Give an example where $|\nu|$ is not finite, and the first assertion does not imply the second.

Theorem 11.5 (Radon-Nikodym).
Let μ be a σ-finite positive measure on the measurable space $(X, \mathcal{M})$ and ν a σ-finite signed measure. Then we can write $\nu = \nu_a + \nu_s$ where ν_a is absolutely continuous w.r.t. μ, and ν_s and μ are mutually singular. Moreover, there exists an extended μ-integrable function f such that

$$\nu_a(E) = \int_E f d\mu.$$

- A function is extended μ-integrable if its negative part is integrable.

Proof: We first prove when μ and ν are both positive and finite measures. Once we have done that, the general case is then not difficult.

We use Hilbert space ideas. Consider the Hilbert space $L^2(X, \rho)$ where $\rho = \mu + \nu$. Consider the map

$$L^2(X, \rho) \ni \psi \mapsto l(\psi) = \int \psi \, d\nu.$$

This is a bounded linear functional, since

$$|l(\psi)| \leq \int |\psi| d\nu \leq \int |\psi| d\rho \leq \rho(X)^{1/2} \|\psi\|_{L^2}$$

using Cauchy-Schwarz. Therefore l is inner product with some element g of $L^2(X, \rho)$:

$$\int \psi \, d\nu = \int \psi g \, d\rho \text{ for all } \psi \in L^2(X, \rho). \tag{11.2}$$

For any measurable set E, with $\rho(E) > 0$, set $\psi = 1_E$. Then we find that

$$\nu(E) = \int 1_E \, d\nu = \int 1_E g \, d\rho,$$

so

$$0 \leq \int 1_E g \, d\rho \leq \rho(E),$$

which implies that $g \leq 1$ a.e. w.r.t. ρ. By changing g on a set of ρ-measure zero, we can assume that $g \leq 1$ everywhere.

Now we define A to be the set where $g < 1$ and B to be the set where $g = 1$. Putting $\psi = 1_B$, we find that

$$\nu(B) = \int 1_B \, d\nu = \int 1_B g \, d\rho = \int 1_B \, d\rho = \nu(B) + \mu(B).$$

Therefore, $\mu(B) = 0$. Since μ is a positive measure, this means that μ is supported in $B^c = A$. So define

$$\nu_a(E) = \nu(E \cap A), \quad \nu_s(E) = \nu(E \cap B).$$

We have just shown that ν_s and μ are mutually singular. Now we show that ν_a is absolutely continuous w.r.t. μ.

First we reformulate Equation (11.2) as

$$\int \psi(1-g) \, d\nu = \int \psi g \, d\mu.$$

It is tempting to try $\psi = (1-g)^{-1}$, which would then give

$$\nu(E) = \int_E d\nu = \int_E (1-g)^{-1}(1-g) \, d\nu = \int_E (1-g)^{-1} g d\mu$$

and the desired conclusion.

However, this is not allowed since $(1-g)^{-1} \notin L^2(X, \rho)$ necessarily. Instead, we approximate, setting

$$\psi = (1 + g + g^2 + \ldots g^n) 1_{E \cap A}$$

which is bounded and therefore in L^2. We obtain

$$\int_{E \cap A} (1 - g^{n+1}) \, d\nu = \int_{E \cap A} g \frac{1 - g^{n+1}}{1 - g} \, d\mu.$$

Since $g < 1$ on A, $1 - g^{n+1} \uparrow 1$ pointwise, so by MCT we get

$$\nu_a(E) = \nu(E \cap A) = \int_{E \cap A} d\nu = \int_{E \cap A} \frac{g}{1-g} \, d\mu.$$

This shows that ν_a is absolutely continuous and we may take $f = g(1-g)^{-1}$, which (by putting $E = X$) the above equation shows is integrable w.r.t. μ.

To prove for σ-finite, positive measures μ, ν, we write X as the disjoint union of a countable family E_j of sets of finite measure. Let μ_j, ν_j be the restrictions of μ, ν to E_j. Then we can decompose ν_j as $\nu_{j,a} + \nu_{j,s}$ as above. Setting $\nu_a = \sum_j \nu_{j,a}$ and $\nu_s = \sum_j \nu_{j,s}$ we satisfy the conditions of the theorem. To treat the case of a signed measure, we treat the positive and negative parts of ν separately. □