Math 3325, 2016

Problem set 1 — discuss in Tutorial on July 25.

1. (a) Show that the set

$$\{f \in L^2([0,4]) \mid \int_0^4 f(x)\, dx = 0\}$$

   is a closed subspace of $L^2([0,4])$.

   (b) Find the closest point in this subspace to the characteristic function of the interval $[0,1]$, in the L^2 metric.

2. Show that the set

$$\{f \in L^2(\mathbb{R}) \mid \lim_{R\to\infty} \int_{-R}^R f(x)\, dx = 0\}$$

   is not a closed subspace of $L^2(\mathbb{R})$.

3. Find the projection of a function f in $L^2([0,1])$ onto the subspace of continuous functions that are linear on $[0, 1/2]$ and $[1/2, 1]$.

4. Consider the function $f = \chi_{[0,1]}$ as an element of $L^1([0,4])$ (note here we look at the L^1 norm, not the L^2 norm). Show that there are lots of functions g that are "halfway between 0 and f", in the sense that $\|g\|_1 = \|f - g\|_1 = 1/2$. On the other hand, show that in a Hilbert space, given $f \neq 0$, there is only one element g satisfying $\|g\| = \|f - g\| = \|f\|/2$, namely $f/2$.

5. Show that the examples of BLTs in lecture 2 are bounded, namely the maps

$$f(x) \mapsto \int_{-\infty}^\infty k(x-y)f(y)\, dy : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

   for $k \in L^1(\mathbb{R})$, and integration on $L^2([0,1])$:

$$f(x) \mapsto \int_0^x f(s)\, ds.$$

6. Let H be a Hilbert space. Suppose that a sequence P_j of orthogonal projections converges to the BLT T in operator norm, i.e. $\|T - P_j\| \to 0$ as $j \to \infty$. Show that T is an orthogonal projection.

7. Suppose that F is a finite rank operator on H. (This means, by definition, that the range of F is finite dimensional.) Show that there exist $x_1, \ldots, x_N, y_1, \ldots, y_N \in H$, where N is the dimension of the range of F, such that

$$Fx = \sum_{j=1}^N (x, x_i) y_i.$$