PROBLEMS IN CATEGORY THEORY — MATH4349 ANU 2016

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Online at: `https://tqft.net/web/teaching/2016/CategoryTheory/questions.pdf`

This is a long list of problems, and many of them are difficult. Certainly, understanding the words in each problem is a necessary first step, and even that may take you some time. I have no expectation that students will attempt *all* of these problems.

I encourage everyone to submit some written work to me every second week: either a solution, or exposition of background material you've been learning to tackling a problem. At the end of the semester I would like everyone to submit a report, e.g. in the 10-30 page range (shorter if you get straight into the technicalities, longer if you include more exposition), on an interesting problem or subject they have encountered through the semester.

1. 'Classical category theory'

There's lots to learn here, and I'd be remiss in not pointing out some highlights. But really I want to get on to the 'fun stuff' — higher categories (tensor categories, braided tensor categories, n-categories, etc) and their relationship with topology and representation theory.

An excellent book to get started is Tom Leinster's Basic Category Theory, soon to be freely available.

I recommend everyone start with Problem 1.2.2..

1.1. **Universal properties.**

1.1.1. What are initial and terminal objects in a category?

1.1.2. What are cones and cocones?

1.1.3. What are limits and colimits?

1.1.4. Give the universal property of the tensor product of vector spaces, and explain that the 'brutal' definition of tensor product, given be picking bases, shows that tensor products exist.

1.2. **Adjoints.** If you like watching maths videos, try Eugenia Cheng's lectures on adjunctions.

1.2.1. What is are left and right adjoints of functors?

1.2.2. Does the functor from graded vector spaces to filtered vector spaces taking $\oplus_n V_n$ to $\cdots \subset W_n \subset W_{n+1} \subset \cdots$ where $W_n = \oplus_{m \leq n} V_m$ have left or right adjoints?

1.2.3. Does the functor π_1 from pointed topological spaces to groups have a left or right adjoint? (Think about it first, and later read [(ht].)

1.2.4. Explain why a category with products and equalizers has all finite limits.

1.2.5. Can you give conditions on a pair of categories $\mathcal{C}$ and $\mathcal{D}$ so that all functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ have left and/or right adjoints? [bro]
1.2.6. There are three possible definitions of adjoints: via Hom-set bijections, via units and counits, and via universal properties. Explain why they are equivalent.

Let me know if you're enjoying this stuff, and I can give you more. (You should probably look up the Yoneda lemma, and try to understand why representable functors are so nice.) But it might be more fun to move along to:

2. Tensor categories

Some general reading on tensor categories and braided tensor categories: [JS91; JS93; Str12; Bae99]. Etingof-Gelaki-Nikshysch-Ostrik have a wonderful book [EGNO15]. Reading the back issues of 'This week's finds in mathematical physics', John Baez's blog (from well before blogs were invented) is an excellent undertaking; perhaps the contents page at `http://math.ucr.edu/home/baez/TWF.html` is the best place to start.

There are too many adjectives applied to monoidal categories [Sel11], and it gets very confusing. A few of the most important: *rigid*, *linear*, *pivotal*, *spherical*. I now tend to write *tensor category* to mean *rigid linear monoidal category*, although this is not yet standard, and explicitly conflicts with the older usage, where tensor and monoidal were synonymous.

A special class of tensor categories (although they are so special often it's best to think of them as an altogether separate beast) are the braided tensor categories. They can be *ribbon*, or *modular*, amongst other things.

For both cases *integral* and *weakly integral* are interesting modifiers, with the intuition being that integrality roughly implies 'coming from finite groups'.

2.1. **Representation theory.**

2.1.1. Why do the representations of a group form a tensor category?
2.1.2. Why do the finite dimensional representations over $\mathbb{C}$ of a finite group form a *finitely semisimple* tensor category.
2.1.3. What about the finite dimensional representations over a finite field?
2.1.4. Why do the representations of a Hopf algebra form a tensor category?
2.1.5. Show that $\mathsf{Rep}\mathfrak{g} \cong \mathsf{Rep}U\mathfrak{g}$, that $U\mathfrak{g}$ is a Hopf algebra, and that this gives the 'usual' tensor product on $\mathsf{Rep}\mathfrak{g}$.

2.2. **Temperley-Lieb.**

2.2.1. What is the Temperley-Lieb category?
2.2.2. Describe the braiding on the Temperley-Lieb category.
2.2.3. Calculate the 3-strand Jones-Wenzl idempotent.
2.2.4. What are the simple objects of the Temperley-Lieb category? (Learn about the 'Karoubi envelope'/'idempotent completion'; you need to do this before the question makes sense.)
2.2.5. Can you show that the Temperley-Lieb category is equivalent to $\mathsf{Rep}U_q\mathfrak{sl}_2$?

2.2.6. Is the Temperley-Lieb category with q a root of unity semisimple? What is the negligible ideal? What does the quotient by the negligible ideal look like? How many simple objects are there?

2.3. Module categories.

2.3.1. The category $\mathsf{Vec}G$ of (finite dimensional) G-graded vector spaces has Vec as a module category. Identify the commutant (or 'dual'). [**MR1976459**]

2.4. Pivotal categories and planar algebras.

2.4.1. Give an example of a strict pivotal category and a non-strict pivotal category.
2.4.2. Explain why any pivotal category is equivalent to a strict pivotal category. [BW99]
2.4.3. What is a planar algebra? [Jon99] How can you build one from a pivotal category, and vice versa? [AMP15, p. 6] and [BP14, §2.1].
2.4.4. Give some examples of finitely presented planar algebras, along with evaluation algorithms that show how to use the presentation to evaluate arbitrary closed diagrams. [MPS10; Pet10; BMPS12]

3. Braided tensor categories and knot invariants

3.0.1. How do you build link invariants out of braided tensor categories?
3.0.2. Can you do the reverse?
3.0.3. Calculate the invariant of the trefoil coming from the braiding in the Temperley-Lieb category. (Can you implement this on a computer?)
3.0.4. Calculate the quantum link invariants of some small links labelled by either the standard representation of $SU(3)$ or the 7-dimensional representation of G_2, using Kuperberg's spiders. [Kup96] (These are equivalently finitely presented planar algebras, in a different language.)
3.0.5. Can you write a program to compute quantum link invariants labelled by the adjoint representation of G_2? (Warning: the formulas in [Kup96] for the adjoint representation are wrong; ask me for the correct ones.)

4. Fusion categories

A fusion category is a *finitely semisimple* tensor category.

4.0.1. What does semisimple mean? (Your answer will involve abelian categories.)
4.0.2. Over $\mathbb{C}$, it suffices to say that there is a collection of objects $\{X_i\}$, such that

$$\mathrm{Hom}(X_i \to X_j) = \begin{cases} \mathbb{C} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and every object is isomorphic to a direct sum of these. Why? [Müg03a, p. 89]

4.0.1. Show that the representation category of a finite group, over $\mathbb{C}$, is a fusion category.

4.1. Rank 2 examples.

4.1.1. How many fusion categories are there with two simple objects 1 and X, such that $X \otimes X \cong 1 \oplus X$?
4.1.2. Are they unitary? Braided?
4.1.3. Can we identify these categories with something coming from representation theory?
4.1.4. What about if we ask for $X \otimes X \cong 1 \oplus nX$? [Ost03]

4.2. Number theory.

4.2.1. Show that the dimension of any object in a fusion category is an algebraic integer.
4.2.2. Show that the dimension of any object in a modular tensor category is a cyclotomic integer.
4.2.3. Show that the dimension of any object in a fusion category is a cyclotomic integer.
4.2.4. Use one of these facts to show that some 'based ring' cannot be the Grothendieck ring of any fusion category.

4.3. Classification.

4.3.1. Explain why there are only finite many fusion categories with global dimension bounded by some constant. (First, show that there are finitely many such based rings.)
4.3.2. What are all the based rings, up to isomorphism, with global dimension at most, say, 10?
4.3.3. How do they categorify as fusion categories?

5. Higher categories

5.1. Field theories.

5.1.1. Explain the classification of $2-1$ dimensional field theories in terms of Frobenius algebras. (What about $2-1-0$ dimensional field theories?) [Mor15] [Fre13, Chapter 23]
5.1.2. Given a pivotal fusion category, the Turaev-Viro construction builds a $3-2-1-0$ dimensional field theory. Calculate the vector spaces and linear maps associated to some simple 2- and 3-manifolds, either in general or for a particular category.
5.1.3. Given a modular tensor category, the Reshetikhin-Turaev construction builds a $3-2-1$ dimensional field theory. Do a calculation!

5.2. Drinfeld centres.

5.2.1. Calculate the Drinfeld centre of the Fibonacci category.
5.2.2. Calculate the Drinfeld centre of $\mathsf{Vec}G$ or $\mathsf{Rep}G$ for some finite groups G.
5.2.3. Show that the Drinfeld centre of a fusion category is equivalent to the representation category of the associated annular category.

5.2.4. For a Levin-Wen topological phase associated to a fusion category $\mathcal{C}$, explain why the 'point-like excitations' are indexed by the Drinfeld centre $Z(\mathcal{C})$.
5.2.5. Show that the Reshetikhin-Turaev invariants based on $Z(\mathcal{C})$ coincide with the Turaev-Viro invariants based on $\mathcal{C}$.

5.3. **Stabilization.**

5.3.1. Explain that monoidal categories are 2-categories with a single 0-morphism.
5.3.2. Explain that braided monoidal categories are 3-categories with a single 0-morphism and a single 1-morphism.
5.3.3. Give other instances of this phenomena.
5.3.4. Prove some special cases of the stabilization hypothesis.
5.3.5. Explain why the Müger centre of a Drinfeld centre is always trivial. [Sny09; Bae99; Müg03b]
5.3.6. Can you generalize this observation?

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