Math 3325, 2017 — Assignment 3

Discuss in tutorial on September 18, and hand in by 5pm on September 29

This assignment is worth 100 marks: 20 for each question below, and 20 for writing quality.

(1) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $f \in L^p(X, \mu)$ be a nonnegative real-valued function. Let $\lambda_f : [0, \infty) \to [0, \infty)$ be the distribution function of f, defined by

$$\lambda_f(\alpha) = \mu\{x \mid f(x) > \alpha\}.$$

(a) (12 marks) First assume that $p = 1$. Show that the integral of f is equal to

$$\int_0^\infty \lambda_f(\alpha)\, d\alpha.$$

Hint: consider the product measure space $(X \times \mathbb{R}, \mathcal{M} \times \mathcal{L}, \mu \times \lambda)$ where $(\mathbb{R}, \mathcal{L}, \lambda)$ is the real line with the σ-algebra of Lebesgue measurable sets and the Lebesgue measure. Relate the integral of f to the set

$$\{(x, \alpha) \in X \times \mathbb{R} \mid 0 \le \alpha \le f(x)\}$$

and use Fubini's theorem.

(b) (8 marks) In a similar fashion, show that for $1 \le p < \infty$

$$\|f\|_p^p = \int_0^\infty p\alpha^{p-1} \lambda_f(\alpha)\, d\alpha.$$

(2) (a) (10 marks) Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and that $(Y, \mathcal{C})$ is a measurable space. Let $F : X \to Y$ be measurable. Prove that the set function $\nu : \mathcal{C} \to [0, \infty]$ given by

$$\nu(E) = \mu(F^{-1}(E))$$

is a measure. That is, show that the pushforward of a measure is indeed a measure, as asserted in the notes.

(b) (10 marks) In this problem we show that the Tonelli theorem can fail if we drop the restriction that the measure spaces are σ-finite.

Let $X = [0, 1]$ with Lebesgue measure m, and let $Y = [0, 1]$ with the discrete σ-algebra $2^{[0,1]}$ and counting measure $\sharp$. Let $f = \chi_{\{(x,x):x\in[0,1]\}}$.

- Explain why $(Y, \sharp)$ is not σ-finite.
- Show that f is measurable in the product σ-algebra.

- Show that
$$\int_X \left(\int_Y f(x,y) d\sharp(y) \right) dm(x) = 1.$$
- Show that
$$\int_Y \left(\int_X f(x,y) dm(x) \right) d\sharp(y) = 0.$$
- Inspired by the fact these integrals are different, construct two different measures on the product σ-algebra with the property that $\mu(E \times F) = m(E)\sharp(F)$ for all E Lebesgue measurable and F an arbitrary subset of $[0,1]$.

(3) (a) (5 marks) Prove a slight strengthening of Fatou's Lemma: if f_n are nonnegative measurable functions, then
$$\liminf \int f_n \geq \int \liminf f_n.$$
Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let a positive number $\epsilon > 0$ be given, and let $E_1, E_2, \ldots$ be a sequence of measurable subsets of X with $\mu(E_i) > \epsilon$ for each i.
For each point $x \in X$, we define $\mathbb{N}_x$ to be the set of positive integers i such that $x \in E_i$. We also define, for any subset J of the positive integers, the upper density to be
$$\limsup_{M \to \infty} \frac{\#(J \cap \{1, \ldots, M\})}{M},$$
where $\#K$ is the number of elements of the set K. The lower density of J is defined similarly, with lim inf replacing lim sup.
(b) (10 marks) Show that there is a subset A of X such that $\mu(A) > 0$, and a $\delta > 0$ so that $\mathbb{N}_a$ has upper density at least δ for every $a \in A$.
Hint: you should find that part (i) — suitably adapted — is helpful in proving part (ii). Of course you can use part (i) in part (ii), even if you were not able to prove (i).
(c) (5 marks) Give an example of X and E_i as above, where the lower density of $\mathbb{N}_x$ is zero for every $x \in X$.

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that a sequence f_n of measurable functions *converges in measure* to f if, for every $\epsilon > 0$,
$$\mu\big(\{x \in X \mid |f_n(x) - f(x)| > \epsilon\}\big) \to 0.$$
This is also called convergence in probability.
(a) (5 marks) Show that if $\|f_n - f\|_{L^1(X)} \to 0$, then f_n converges to f in measure.

(b) (5 marks) Give an example to show that the converse is false.

(c) (10 marks) Suppose that $(X, \mathcal{M}, \mu)$ is a measure space, (f_n) is a sequence of real-valued measurable functions on X such that all f_n are dominated by a fixed integrable function g. If f_n converges to f in measure, show that f is integrable and that

$$\int f_n \to \int f.$$

That is, DCT holds with the hypothesis of pointwise a.e. convergence replaced by convergence in measure.