**Math 3325, 2017 — Assignment 2**

**Discuss in tutorial on August 28, and hand in by 5pm on Friday September 1**

This assignment is out of 100: 4 questions worth a total of 80 marks, and 20 marks for your writing quality. Acknowledge any help that you receive, either from a book, another student, or an internet source. Discussion of the problems with other students is allowed, but you must write your solutions yourself. Do not look at anyone else's solutions, and do not show your solutions to another student.

(1) Note: attempt part (iii) even if you cannot do parts (i) and (ii).

(i) (8 marks) Suppose that $f_n$ and $f$ are functions in $L^1(\mathbb{R}^n)$, such that $f_n \to f$ in $L^1$. Show that $\hat{f}_n \to \hat{f}$ uniformly. Here the Fourier transform is defined by the convergent integral

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

(ii) (8 marks) Suppose that a sequence of bounded continuous functions $g_n$ converge uniformly to the function $h$, and in $L^2$ to the function $k$. Show that $h = k$ a.e.

(iii) (6 marks) Now suppose that $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then we have two, potentially different, definitions of the Fourier transform of $f$: firstly by a convergent integral,

$$\mathcal{F}_1 f(\xi) = \int e^{-ix\cdot\xi} f(x)\, dx$$

and secondly by taking a sequence of Schwartz functions $f_n$ converging to $f$ in $L^2$, and taking the $L^2$ limit of the sequence $\mathcal{F}_1 f_n$ (call this limit $\mathcal{F}_2 f$). Use parts (i) and (ii) to show that $\mathcal{F}_1 f = \mathcal{F}_2 f$ a.e. Suggestion: use the fact, proved in the text, that there is a sequence of Schwartz functions $f_n$ converging to $f$ in both the $L^1$ and the $L^2$ sense.

(2) Consider the following initial value problem: find a function $u(x, t)$, defined on

$$\{(x, t) \in \mathbb{R}^3 \times \mathbb{R} \mid t \geq 0\},$$

satisfying

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2},$$

with initial conditions

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = 0$$

at time $t = 0$. We assume that $f$ is a Schwartz function of $(x_1, x_2, x_3)$.

(i) (5 marks) Using the Fourier transform in the $x_i$ variables only, find an expression for the (partial) Fourier transform of $u$, that is $\hat{u}(\xi, t)$, in terms of the Fourier transform of $f$.

(ii) (5 marks) *Briefly* explain why $u(\cdot, t)$ is a Schwartz function on $\mathbb{R}^3$ for each fixed $t$.

(iii) (5 marks) Show that for each $t \geq 0$, we have

$$\|u(\cdot, t)\|_{L^2(\mathbb{R}^3)} \leq \|f\|_{L^2(\mathbb{R}^3)}.$$

(iv) (5 marks) Show that for each $t \geq 0$, we have

$$\Big\| \frac{\partial u}{\partial t}(\cdot, t) \Big\|^2_{L^2(\mathbb{R}^3)} + \sum_{i=1}^{3} \Big\| \frac{\partial u}{\partial x_i}(\cdot, t) \Big\|^2_{L^2(\mathbb{R}^3)} = \sum_{i=1}^{3} \Big\| \frac{\partial f}{\partial x_i} \Big\|^2_{L^2(\mathbb{R}^3)}.$$

(3) Suppose that $g \in L^1(\mathbb{R})$ is compactly supported.

(i) (5 marks) Show that $\hat{g}(\xi)$ is $C^\infty$.

(ii) (5 marks) Show that $\hat{g}(\xi)$ is analytic and entire. That is, show that the Taylor series of $\hat{g}$ at $\xi = 0$ has infinite radius of convergence, and converges to $\hat{g}$.

(iii) (5 marks) Show that the only $f \in L^1(\mathbb{R})$ satisfying $f = f * f$ is the zero function.

(iv) (5 marks) Find a nonzero $g \in L^2(\mathbb{R})$ such that $g = g * g$.

(4) (i) (10 marks) Show that bounded continuous functions are not dense in $L^\infty(\mathbb{R})$.

(ii) (10 marks) Show that if functions $f, k$ are in $L^1(\mathbb{R}^n)$ then $\|f * k\|_{L^1} \leq \|f\|_{L^1} \|k\|_{L^1}$. Use this, and a density argument, to show that if $\phi \in C_c^\infty(\mathbb{R}^n)$ is nonnegative with integral 1, then

$$f * \phi_\epsilon \to f \text{ in } L^1(\mathbb{R}^n).$$

(Here $C_c^\infty(\mathbb{R}^n)$ denotes the compactly supported smooth functions, and, as in the notes, $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$.)