**Math 3325, 2017 — Assignment 4**

**Hand in by 5pm on October 27**

This assignment is worth 100 marks: 80 for the questions below, and 20 for writing quality. (My apologies for the initially nonsensical grading scheme. I have now reorganised the assignment into 5 questions, each worth 20 marks. I will count your best 4 out of 5 questions, and then give 20 marks for writing quality.)

(1) (i) (4 marks) Let $A$ be a bounded self-adjoint operator. Show that $U = (A - iI)(A + iI)^{-1}$ is unitary.
  (ii) (4 marks) Let $X$ be a bounded operator. Show that $\sigma(X^*) = \{\bar{\lambda} | \lambda \in \sigma(X)\}$ and that if $X$ is invertible then $\sigma(X^{-1}) = \{\lambda^{-1} | \lambda \in \sigma(X)\}$.
  (iii) (4 marks) Show that the only positive unitary operator is $I$.
  (iv) (4 marks) Let $A$ be a bounded self-adjoint operator. Show that $A \geq kI$ for $k \in \mathbb{R}$ if and only if $\sigma(A) \subset [k, \infty)$.
  (v) (4 marks) Let $A$ be a bounded self-adjoint operator. Show that if $A \geq I$, $A^n \geq I$ for every positive integer $n$.

(2) Let $S$ be a linear subspace of $C([0,1])$. Since $C([0,1])$ is a subset of $L^2([0,1])$ we can also regard it as a subspace of $L^2([0,1])$. We assume that $S$ is closed as a subspace of $L^2([0,1])$, i.e., in the $L^2$ topology.
  (i) (5 marks) Show that $S$ is a closed subspace of $C([0,1])$ (under the sup norm).
  (ii) (5 marks) Show that there exists $M > 0$ such that for all $f \in S$,

$$\|f\|_2 \leq \|f\|_\infty \leq M\|f\|_2.$$

  (Use the closed graph theorem.)
  (iii) (5 marks) Fix $y \in [0,1]$. Show that there exists a function $k_y \in L^2([0,1])$, with $\|k_y\|_{L^2([0,1])} \leq M$, such that

$$f(y) = \int_0^1 k_y(x) f(x)\, dx$$

  for all $f \in S$. (Use the Hilbert space Riesz representation theorem.)
  (iv) (5 marks) Show that the $L^2$ unit ball of $S$ is compact, and hence that $S$ finite dimensional. (Show that a sequence in the unit ball which converges weakly converges in norm.)

(3) Let $p \in (1, \infty)$. Let $l^p$ denote the Banach space of $p$-summable sequences of complex numbers, and let $e_i$ denote the element of $l^p$ with $j$th entry equal to 0 for $j \neq i$, and 1 for $j = i$.

(i) (5 marks) Show that a sequence $(x_n)$ in $l^p$, where $x_n = (a_n^m)_{m=1}^\infty$, converges weakly to zero iff the norms $\|x_n\|$ are uniformly bounded, and $a_n^m$ converges to 0 in $\mathbb{C}$ for each fixed $m$ as $n \to \infty$.

Define the set $F \subset l^p$ by

$$F = \{e_m + m e_n \mid m < n,\ m, n \in \mathbb{N}\}.$$

(ii) (5 marks) Show that $F$ is closed in the strong topology.

(iii) (5 marks) Show that 0 is in the closure of $F$ in the weak topology.

(iv) (5 marks) Show that there is no sequence contained in $F$ that converges weakly to zero. (Use the result of part (a)). Remark: parts (c) and (d) shows that the weak topology is not metrizable, since for a metrizable topology, the closure of a set is precisely the set of limit points of convergent sequences from that set.

(4) (i) (8 marks) Suppose that $X$ is a Banach space, and that $x_i$ is a sequence in $X$ converging weakly to $x$. Show that

$$\|x\| \leq \limsup_{i \to \infty} \|x_i\|.$$

(ii) (6 marks) Suppose that $X$ and $Y$ and Banach spaces, and that $T : X \to Y$ is a bounded linear transformation. Show that $T$ is also continuous if both $X$ and $Y$ are given the weak topology.

(iii) (6 marks) Let $\phi \in C_c^\infty([-1, 1])$ be a smooth, compactly supported function with integral 2. Consider the sequence of measures on $[-1, 1]$:

$$\mu_n = n\phi(nx)dx,$$

where $dx$ is Lebesgue measure. These measures converge in the weak-$*$ topology. What is the limit measure?

(5) (i) (6 marks) Let $(A_n)$ be a decreasing sequence of nonempty closed balls in a Banach space. Show that the intersection of the $A_n$ is nonempty. (Do not assume that the radii converge to zero.)

(ii) (8 marks) Let $(B_n)$ be a decreasing sequence of closed, bounded, nonempty convex sets in a reflexive Banach space $Y$. Show that the intersection of the $B_n$ is nonempty. (Hint: first show that the sets $B_n$ are weakly closed, using the Separating Hyperplane theorem. If you can't, assume it and complete the rest of the problem, using Banach-Alaoglu.)

(iii) (6 marks) Let $X = L^1(\mathbb{R})$. Find a decreasing sequence of closed, bounded, nonempty convex sets $C_n \subset X$ whose intersection is empty.