3 Compact operators on Hilbert space

There is a class of bounded linear transformation on a Hilbert space $H$ that is closely analogous to linear transformations between finite-dimensional spaces—the compact operators. Throughout, we will take $H$ to be separable and infinite dimensional. Recall that there is only one such $H$ up to unitary equivalence.

Let us define the closed unit ball $B \subset H$ to be

$$B = \{ f \in H | \|f\| \leq 1 \}.$$

Notice that $B$ is not compact. Indeed, if $e_1, e_2, \ldots$ is an orthonormal basis of $H$, then this is a sequence in $B$ with no convergent subsequence, since $\|e_i - e_j\| = \sqrt{2} \neq 0$.

**Definition 3.1.** A bounded linear transformation $T : H \to H$ is compact if the closure of $T(B)$ is compact in $H$. Equivalently, $T$ is compact if, for every bounded sequence $f_1, f_2, \ldots$ in $H$, $T(f_n)$ contains a convergent subsequence.

Thus, the identity operator on $H$ is not compact. Here are some examples of compact operators:

- **Finite rank operators.** A bounded linear transformation is said to be of finite rank if its range is a finite dimensional. Let $F$ be a finite rank bounded linear transformation. Then $F(B)$ is a bounded set contained in a finite dimensional subspace of $H$. Its closure is therefore compact (since closed, bounded subsets of $\mathbb{C}^n$ are compact).

**Exercise.** If $F$ is finite rank, let $n = \dim \text{ran}(F)$. Show that there are $n$ vectors $f_1, f_2, \ldots, f_n$ so that, with $\mathcal{S} = \{f_1, \ldots, f_n\}$, $H = \text{span}(\mathcal{S}) \oplus \mathcal{S}$. Hence, show that $F$ has the form

$$Fk = \sum_{j=1}^{n} g_j(f_j, k)$$

for some vectors $g_j$.

- **Integral operators.** If $H = L^2([0,1])$, let the operator $T$ be defined by

$$T(f)(x) = \int_0^1 K(x,y)f(y)\,dy.$$

Then if $K(x,y)$ is $L^2$ on $[0,1]^2$, then $T$ is compact. We will show this shortly.

3.1 Properties of compact operators

**Proposition 3.2 (Proposition 6.1 of [SS]).** Let $T$ be a bounded linear operator on $H$. If $T$ is compact, then $ST$ and $TS$ are compact.

(i) Suppose that there exists a sequence $T_n$ of compact operators such that $\|T - T_n\| \to 0$ as $n \to \infty$. Then $T$ is compact.

(ii) Every compact operator $T$ is the norm limit of a sequence of finite rank operators. Theorem 3.1 in [SS].

**Remarks on the proof:**

- (i) uses some standard point-set topology.
- (ii) uses a diagonal argument.
- (iii) is proved using a family of projection operators associated to an ONB.
- (iv) follows readily from (iii).

**Proof.** The proof of (i) is straightforward. Let $(T_n)$ be a bounded sequence. Then $T_nF$ is another bounded sequence, and hence $T_nF$ has a convergent subsequence. Hence $T$ is compact. Also, we note that $T_nF$ has a convergent subsequence $T_mF$, and since $T$ is continuous, $T_mF$ is convergent. Therefore $T$ is compact.

**Exercise.** Let $f$ be a bounded sequence. Then since $T$ is compact, there is a subsequence $f_{j_k}$ such that $T f_{j_k}$ converges. Since $T$ is compact there is a subsequence $f_{j_{m_k}}$ of $f_{j_k}$ such that $T f_{j_{m_k}}$ converges. And so on, we thus generate a family of nested subsequences $f_{j_{m_{k}}}$.

For example, let $f_1, f_2, \ldots$ be a bounded sequence. Then $f_{j_k}$ converges. Let $f_{j_k} = f_{j_{m_k}}$. Then $f_{j_{m_k}}$ is eventually a subsequence of the $i$th subsequence $f_{j_{m_{k}}}$, and so on for each $n$. We now claim that $T$ is the Cauchy sequence, and hence convergent. To see this we write

$$|T(y_n - y)| \leq |T - T_n|(y_n - y) + |T_n y_n - T_n y| + |T_n y - T y|$$

which is valid for any $n$. Let $M$ be an upper bound on the $|y_n|$. Then the first and third terms are bounded by $M|T - T_n|$, which is small provided $n$ is chosen large enough. Fixing any sufficiently large $n$, the second term is small if $k$ is large enough.

(ii) Choose an orthonormal basis $e_1, e_2, \ldots$ and let $P_n$ be the orthogonal projection onto the span of the first $n$ basis vectors, and $Q_n = I - P_n$. Then $(Q_nTf)$ is a nonincreasing function of $n$, so therefore $(Q_nTf)$ is nonincreasing in $n$. If $(Q_nTf) \to f$, then the statement is proved, so assume, for a contradiction, that $(Q_nTf) \not\to f$ for all $n$. Choose $f_n, |Q_nf_n| = 1$, such that $\|Q_nTf_n\| > c/2$ for each $n$. By compactness of $T$, there is a subsequence such that $T f_{j_{m_k}} \to g$ as $k \to \infty$. Then $Q_kQ_n f_{j_{m_k}} \to Q_kQ_n f$ as $k \to \infty$. Hence $f_n \to f$.

**Exercise.** Let $K(x, y)$ be a continuous function on $[a, b] \times [a, b]$. Show that the integral operator on $L^2([a, b])$

$$f(x) = \int_a^b K(x, y)f(y)\,dy$$

is self-adjoint exactly if $K(x, y) = K(y, x)$. If $K(x, y)$ is only measurable and measurable, then the same result holds for a.e. $(x, y)$.

It turns out that for self-adjoint compact operators, the diagonal example above is in fact the general case.

**Theorem 3.3.** Let $T$ be a compact self-adjoint operator on $H$. Then there is an orthonormal basis $e_1, e_2, \ldots$ of $H$ consisting of eigenvectors of $T$. Thus $T e_i = \lambda_i e_i$ and we have $\lambda_i \in \mathbb{R}$ and $\lambda_i \to 0$ as $i \to \infty$.

- This is the analogue in infinite dimensions of the fact that a real symmetric matrix is diagonalizable via an orthogonal matrix.

Steps in the proof:

1. Show that $|T| = \sup_{f \neq 0} \|Tf\|$.

2. Show that the quantity on the RHS takes a maximum value at some $f$ which is an eigenvector of $T$.

3. Eigenspaces of $T$ corresponding to distinct eigenvalues are orthogonal.

4. The operator $T$ restricts to a compact self-adjoint operator $T |_{V}$, whenever $V$ is an eigenspace, or direct sum of eigenspaces.

5. Thus the direct sum of all eigenspaces must be the whole space.

**Proof.** 1. Claim: For any self-adjoint operator $T$ (compact or not),

$$|T| = \sup_{f \neq 0} \|Tf\|.$$

To see this, we use the characterization

$$|T| = \sup_{f \neq 0} \|Tf\|.$$
So, with \( M = \sup_{\|f\| = 1} |(Tf, f)| \), we have \( \|T\| \geq M \). To prove \( \|T\| \leq M \), we write using the self-adjointness of \( T \):

\[
4 \text{Re}(Tf, g) = (T(f + g), f + g) - (Tf, f) - (Tg, g).
\]

Then, we get

\[
4|\text{Re}(Tf, g)| \leq M(\|f + g\|^2 + \|f - g\|^2),
\]

and the parallelogram law gives

\[
\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2) = 4.
\]

So \( |\text{Re}(Tf, g)| \leq M \). Replacing \( g \) by \( \psi \sqrt{\|\psi\|^2} \), we can make \( |\text{Re}(Tf, g)| = |\langle Tf, g \rangle| \) and the proof is complete.

2. Therefore, either \( T = 0 \), in which case the theorem is trivial, or \( |\langle Tf, f \rangle| > 0 \) for some \( f \) with \( \|f\| = 1 \). By replacing \( T \) with \( -T \) if necessary, we can assume that there exists \( f \) with \( \langle Tf, f \rangle > 0 \) (note that by self-adjointness, \( \langle Tf, f \rangle \) is real for all \( f \)).

Consider the problem of maximizing \( \langle Tf, f \rangle \) as \( f \) ranges over the unit ball of \( H \). By 1, the set of values \( \langle Tf, f \rangle \), \( f \in H \), has a supremum \( \|T\| > 0 \), so we may take a sequence \( f_n, \|f_n\| = 1 \) with

\[
\langle Tf_n, f_n \rangle \to \mu.
\]

I claim that

\[
\|f_n - \mu f\|_A \to 0.
\]

To see this, we square the LHS and compute

\[
0 \leq \|f_n - \mu f\|_A^2 = \|Tf_n - \mu f\|_A^2 = \|Tf_n - \mu f\|_A^2 - 2\Re (\langle Tf_n, f_n \rangle) \mu + \mu^2 \leq 2\|f_n - f\|_0 (\|f_n\|_0 - \mu) \to 0,
\]

which verifies the claim. Now we exploit compactness of \( T \): the sequence \( \{Tf_n\} \) has a subsequence converging, say to \( hf \). Passing to the subsequence we may assume that the sequence \( \{Tf_n\} \) itself converges. Then \( f_n \) converges to \( f \), since

\[
\|f_n - f\|_0 \leq \mu^{-1} \|Tf_n - \mu f\|_A \to 0.
\]

By continuity of \( T \), \( Tf = \lim_n T(f_n) = \mu f \). Thus we have found an eigenvector \( f \) of \( T \).

3.4 Sturm-Liouville operators

A Sturm-Liouville operator is an operator \( L : C^2([a,b]) \to C([a,b]) \) of the form

\[
L(f) = q(x)\frac{d^2}{dx^2}f(x) + p(x)f(x) \quad (a < x < b)
\]

where \( q(x) \) is a continuous function. Here we will assume that \( q(x) \geq 0 \). We will prove that there is a complete set of eigenfunctions of \( L \) in \( L^2([a,b]) \), that is, functions \( \phi_n(x) \) such that

\[
L\phi_n(x) = \lambda_n \phi_n(x).
\]

Notice that if \( q(x) \equiv 1 \), and \( [a,b] = [0,\pi] \), then a complete set of eigenfunctions is the set \( \sin(nx) \), \( n = 1, 2, \ldots \). The result can then be viewed as a generalized, ‘variable coefficient’ version of Fourier series.

As before, the operator \( L \) cannot be bounded on \( L^2 \), since it involves derivatives. The idea is to construct the inverse operator to \( L \). This can be done in a surprisingly explicit way. What we do is look for two solutions \( \phi_1(x) \) and \( \phi_2(x) \) of the equation \( Lf = 0 \). These are specified by their initial conditions: we require that \( \phi_1(a) = \phi_2(a) = 0 \), \( \phi_1'(a) = 1 \), while \( \phi_2(a) = 0 \), \( \phi_2'(a) = 1 \). I claim that \( \phi_1(b) \neq 0 \). Otherwise, compute

\[
0 = \int_a^b \phi_1(x)(\phi'_1(x) - \phi'_2(x))dx = \int_a^b \phi_1(x)(-\phi''_2(x) + q(x)\phi(x) - \phi''_1(x) + q(x)\phi(x))dx = \int_a^b \phi'(x)^2 + q(x)\phi(x)^2 dx.
\]

Here we integrated by parts and used the boundary conditions \( \phi_1(a) = \phi_2(a) = 0 \) to eliminate the boundary term (which is \( \phi_1(b)\phi_2(b) - \phi_2(a)\phi_1(a) \)). Because we assumed that \( q(x) \geq 0 \), this can only be \( 0 \) if \( \phi_1 \) is identically zero, which contradicts the condition \( \phi_1(a) = 1 \).

We next conclude that \( \phi_1 \) and \( \phi_2 \) are linearly independent; otherwise \( \phi_2(b) = 0 \).

Recall from ODE theory that the Wronskian,

\[
W(x) = \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x)
\]

is constant in \( x \). Evaluating at \( x = b \) we see that it is nonzero. We write \( W = W(b) \).

Now I claim that the integral operator \( T \) with kernel

\[
K(x,y) = \phi_1(x)\phi_2(y)/W, \quad x \leq y \quad \phi_1(x)\phi_2(y)/W, \quad x \geq y
\]

is an bounded operator on \( L^2([a,b]) \). An interesting computation shows that, for all continuous \( f \in C([a,b]), \)

\[
L(f) = \phi_1(x)f(x)/W - f(x)(\phi'_1(x)/W + \phi'_2(x)/W) = \phi_1(x)(f(x)/W - \phi_2(x)\phi_1'(x)/W - \phi_1(x)\phi_2'(x)/W).
\]

as elements of \( L^2([a,b]) \), form an ONB. We can do this by manufacturing a compact self-adjoint operator \( T \) for which these functions are the eigenfunctions! Which operator? You might think of \( T = -d/dx \), but this doesn’t work because it is not bounded, let alone compact. Instead, we use integration by parts.

Check that the operator

\[
f(0) = \int_a^b f(x) dx - \int_a^b (\phi'_1(x)/W + \phi'_2(x)/W) dx
\]

is compact and self-adjoint, and that its eigenfunctions are precisely the set \( \{2\pi^{1/2}e^{i(n+1/2)\theta}, n \in \mathbb{Z}\} \).

Example. Be warned: the situation for non-self-adjoint compact operators is quite different. For example, consider the operator \( T \) where \( T \) maps \( e_i \) to \( e_j \) and \( U \) maps \( e_i \) to \( e_{i+1} \). This is compact, but it has no eigenvectors at all.

3.3 Applications of the spectral theorem

There are many applications of this result. One I want to mention here is to showing that orthonormal sets are actually bases. For example, suppose we want to show that the orthonormal set of functions

\[
\{2\pi^{-1/2}e^{i(n+1/2)\theta}, \quad n \in \mathbb{Z}\}
\]

is an orthonormal set of functions on \( L^2([a,b]) \). An interesting computation shows that, for all continuous \( f \in C([a,b]), \)

\[
L(f) = \phi_1(x)f(x)/W - f(x)(\phi'_1(x)/W + \phi'_2(x)/W)
\]

is a bounded operator on \( L^2([a,b]) \). An interesting computation shows that, for all continuous \( f \in C([a,b]), \)

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\[
L(f) = \phi_1(x)f(x)/W - f(x)(\phi'_1(x)/W + \phi'_2(x)/W)
\]
This vanishes, for example, if we require that $f$ and $g$ vanish at $a$ and $b$. (Another suitable condition is that $f'$ and $g'$ vanish at $a$ and $b$.)