7 Dirichlet Problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, where $n \geq 2$, and let $f : \partial\Omega \to \mathbb{R}$ be a continuous function. The Dirichlet problem is to find a continuous function $u : \overline{\Omega} \to \mathbb{R}$ such that u is C^2 in Ω, with

$$\Delta u = 0 \text{ in } \Omega \text{ and } u|_{\partial\Omega} = f.$$

That is, u is harmonic in Ω and continuous up to the boundary, with boundary values f. Physically, this would represent the equilibrium temperature distribution in a region with fixed temperature applied at the boundary of the region.

7.1 Dirichlet's principle

There is an informal idea called Dirichlet's principle for solving this problem. Consider all functions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ taking the boundary values f. Choose the one that minimizes the 'energy'

$$E(u) = \int_\Omega |\nabla u(x)|^2 \, dx. \tag{7.1}$$

Then, for any C^2 function v on Ω, with zero boundary values,

$$\frac{d}{dt} E(u + tv) = 0.$$

The derivative at $t = 0$ is

$$2 \int_\Omega \sum_i \nabla_i u(x) \nabla_i v(x) \, dx.$$

Integrating by parts which is valid since v has zero boundary values, we find that

$$0 = \int (\Delta u(x)) v(x) \, dx = 0.$$

But this implies that $\Delta u = 0$. So the minimizing function is automatically harmonic.

7.2 Problems with Dirichlet's principle

There are two problems with this as a 'proof'. One is that we need to show that a minimizer of the energy $E(u)$ exists. Existence could fail in two possible ways:

- First, there might be no functions that satisfy the boundary condition and such that $E(u)$ is finite (let us call these *admissible* functions) — an example is given in the text on p233. We shall deal, at least initially, with this problem by defining it away: we shall assume initially that there is a C^1 function F, defined in a neighbourhood of Ω, so that $f = F|_{\partial\Omega}$. Then F restricted to $\overline{\Omega}$ is a function with $E(F) < \infty$ satisfying the boundary condition, so there is a nonempty set of admissible functions over which to minimize the energy.
- Given this, the set of energies of admissible functions clearly has an infimum. However, that infimum may not be achieved. To deal with this, we need a condition on the boundary of the domain.

7.3 The strategy

To introduce Hilbert space methods, we use (7.1) as a Hilbert space norm (squared) on a suitable function space.

To define this Hilbert space, we start with the space of C^1 functions on Ω with sesquilinear form

$$\langle U, V\rangle = \int_\Omega \nabla U(x) \cdot \overline{\nabla V}(x)\, dx \tag{7.2}$$

which makes sense for all $U, V \in C^1(\overline{\Omega})$, for example. This satisfies all the conditions of an inner product, except for strict positivity, which requires that $\langle U, U\rangle = 0 \implies U = 0$. However, here $\langle U, U\rangle = 0$ only implies that U is constant. So we take the quotient $\mathcal{H}_0$ of $C^1(\overline{\Omega})$ by constant functions; on this space, we have a genuine inner product; hence, $\mathcal{H}_0$ is a pre-Hilbert space (satisfying every property of Hilbert spaces except completeness).

Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$. This is a Hilbert space. It is perhaps not obvious what the elements of $\mathcal{H}$ actually 'are', since they are formally equivalence classes of Cauchy sequences of elements of $\mathcal{H}_0$. In fact, we only need to know about the subspace of elements of $\mathcal{H}$ that vanish at the boundary.

Proposition 7.1 (S&S Lemma 4.9 ff.).

(i) Suppose that $v \in C^1(\overline{\Omega})$ and vanishes at $\partial\Omega$. Then

$$\int_\Omega |v(x)|^2\, dx \le \frac{1}{n}(\operatorname{diam}\Omega)^2 \int_\Omega |\nabla v(x)|^2\, dx.$$

(ii) Let S be the closure in $\mathcal{H}$ of the subspace of functions in $C^1(\Omega)$ vanishing at $\partial\Omega$. Then elements of S are L^2 functions.

Remark. This is not claiming (and it is not true) that every $w \in L^2(\Omega)$ is an element of S. In fact, elements of S are 'somewhat smoother' than the typical element of $L^2(\Omega)$. S is a 'Sobolev space', denoted $W_0^{1,2}(\Omega)$ or $H_0^1(\Omega)$.

Generally, $W^{k,p}$ is the set of L^p functions, all of whose $\leq k$-th order partial derivatives are also in L^p. The subscript zero in $W_0^{1,2}$ denotes those functions which additionally vanish on the boundary.

Proof: We first note that if $f \in C^1(I)$ for some real interval $I = [a, b]$, and f vanishes at one endpoint of I, say at a, then

$$\int_I |f(t)|^2 \, dt \leq |I|^2 \int_I |f'(t)|^2 \, dt$$

which is an application of the fundamental theorem of calculus and Cauchy-Schwarz.

We have $\int |f(t)|^2 \, dt \leq |I| \, (\max |f(t)|)^2$. Now

$$\begin{aligned} f(t) &= \int_a^t f(s) ds \\ &\leq \int_a^t |f(s)| ds \\ &\leq \int_I |f(s)) ds. \end{aligned}$$

Thus

$$\begin{aligned} \int |f(t)|^2 \, dt &\leq |I| \int_I |f'(s)| ds \\ &\leq |I||I| \int_I |f'(s)|^2 ds \end{aligned}$$

by Cauchy-Schwarz.

To prove (i), write $x \in \mathbb{R}^n$ as $x = (x_1, x')$ and fix x'; then $\Omega_{x'} = \{x_1 \mid (x_1, x') \in \Omega\}$ is a disjoint union of open intervals I_j. For each one, we have

$$\int_{I_j} |v(x_1, x')|^2 \, dx_1 \leq |I_j|^2 \int_{I_j} |\nabla_1 v(x_1, x')|^2 \, dx_1$$

and summing over j, we get (since $|I_j| \leq \operatorname{diam} \Omega$),

$$\int_{\Omega_{x'}} |v(x_1, x')|^2 \, dx_1 \leq (\operatorname{diam} \Omega)^2 \int_{\Omega_{x'}} |\nabla_1 v(x_1, x')|^2 \, dx_1.$$

Integrating over x', and then performing the same estimate in each of the coordinate directions and then summing, gives (i).

(ii) Let (v_n) be a sequence in $C^1(\overline{\Omega})$ all of which vanish at $\partial\Omega$, which is Cauchy in the norm $\|v_n\|_{\mathcal{H}} = \sqrt{\langle v_n, v_n\rangle}$ defined in (7.2). Then, by (i), the sequence is also Cauchy in the L^2 norm, so they converge to an L^2 function v. The element in $\mathcal{H}$ corresponding to the Cauchy sequence (v_n) can be unambiguously identified with v. $\square$

Now, recall that we are assuming the boundary data f is the restriction to $\partial\Omega$ of a function $F \in C^1(\overline{\Omega})$. Consider the space of all functions in $\mathcal{H}_0$ with boundary data f. This is precisely the space of functions equal to the sum of F and a function with zero boundary data. That is, it is the translate $S_0 + F$ of the subspace S_0 of $C^1(\overline{\Omega})$ functions with zero boundary data by the function F. Let S be the closure of S_0 in $\mathcal{H}$. Then the closure of $S_0 + F$ is $S + F$.

Consider the problem of finding the element u of $S + F$ of minimum norm. By translating, it is equivalent to finding the element of S closest to the element $-F \in \mathcal{H}$. As we have seen, there is a unique such element, namely the orthogonal projection $P_S(-F)$ onto S applied to $-F$. Translating back again, let

$$u = F - P_S(F).$$

This function u is our candidate for the solution to the Dirichlet problem.

There are still two major problems with this:

• We don't know that u is harmonic. In fact, a priori, u is only L^2, so we don't even know that u is differentiable, let alone that it satisfies a second order PDE.

• We also don't know that u attains the boundary data f. And in fact, it need not, without extra conditions on the boundary.

It turns out, however, that the Hilbert space formalism allows us to resolve these questions.

In what follows, I'll state the results that solve these problems, but not go through all the proofs, which appear at the end of the notes. (You can read these if you like; they are good examples!)

7.4 Weakly harmonic functions

Let us say that an L^2 function u is weakly harmonic in Ω if

$$\int_\Omega u(x)(\Delta\psi(x))\,dx = 0$$

for all C^∞ functions ψ supported in Ω.

(Notice that this support condition means that the support of ψ is a positive distance from $\partial\Omega$, since by assumption the support of Ω is compact, and contained in the open set Ω.)

If u happens to be smooth, then we can integrate by parts twice and we see that u is harmonic in the usual sense, i.e. twice continuously differentiable and satisfying $\Delta u = 0$ at each point.

Lemma 7.2. *Our candidate solution u is weakly harmonic.*

Proof: Since $u = F - P_S(F)$, u is orthogonal to S. Let u_n be a sequence of elements of $S_0 + F$ converging to u in the norm of $\mathcal{H}$. Then, $\langle u_n, \psi\rangle \to 0$ for all $\psi \in S$, in particular for all $\psi \in C_c^\infty(\Omega)$. Now

$$\begin{aligned}\langle u_n, \psi\rangle &= \int_\Omega \nabla u_n(x) \cdot \nabla\psi(x)\, dx \\ &= -\int_\Omega u_n(x)(\Delta\psi(x))\, dx \to 0.\end{aligned}$$

But $u_n - F \to u - F$ in S, implying that $u_n - F \to u - F$ in L^2, and hence $u_n \to u$ in $L^2(\Omega)$. Therefore

$$\int_\Omega u(x)\Delta\psi(x)\, dx = 0$$

for all $\psi \in C_c^\infty(\Omega)$, i.e., u is weakly harmonic. □

In fact, we can replace u with a smooth u that is actually harmonic:

Theorem 7.3. *Suppose that $u \in L^1_{loc}(\Omega)$ is weakly harmonic in Ω. Then u can be modified on a set of measure zero so that it becomes C^∞ and harmonic in Ω.*

(Here L^1_{loc} denotes the locally integrable Stone-Weierstrass functions, i.e. those functions so that for every compact $K \subset \Omega$, the restriction to K is an L^1 function.)

This follows from the following very important characterization of harmonic functions:

Proposition 7.4. *Suppose that u is C^2 and harmonic. Then it has the mean value property: for each $x \in \Omega$ and each $r > 0$ such that $\overline{B(x,r)} \subset \Omega$, we have*

$$\begin{aligned}u(x) &= \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\, dy \\ &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(\theta) d\sigma(\theta).\end{aligned}$$

Conversely, if u is C^2 and satisfies the mean value property, then u is harmonic.

We know our function u is in L^2, but does it have finite energy? This should be the case if we believe Dirichlet's principle.

Lemma 7.5. *The function u satisfies*

$$\int_\Omega |\nabla u(x)|^2 \, dx < \infty.$$

7.5 Boundary values

Does our proposed u actually have the desired boundary values f?

To simplify matters let's assume that $n = 2$ and that the domain is convex. That is, for every point $x \in \partial\Omega$, there is a line π through x so that Ω is on one side of π. Under this condition, let us show that u is continuous up to the boundary and attains the boundary value f.

For $z \in \Omega$, write $\delta(z)$ for the distance from $\partial\Omega$. Let $\mathrm{Av}(g)(z)$ denote the mean value of the function g over the ball $B(z, \delta(z)/2)$.

The key technical point is:

Lemma 7.6. *Assume that Ω is convex. Let $v \in C^1(\overline{\Omega})$, and zero on the boundary. Then*

$$\mathrm{Av}(v)(x) \leq \frac{16}{\pi} \int_{B(x,2\delta(x))\cap\Omega} |\nabla v(x)|^2 \, dx. \tag{7.3}$$

Lemma 7.7. *Equation* (7.3) *applies to the function $F - u$, even though it isn't C^1.*

Proof: The function $F - u$ lies in the subspace S which is the *closure* of C^1 functions vanishing at the boundary. Let v_n be a sequence of such functions converging in the $\mathcal{H}$ norm to $F - u$. Then $\|v_n - (F - u)\|_{\mathcal{H}} \to 0$ and $\|v_n - (F - u)\|_{L^2} \to 0$. So applying equation (7.3) to each v_n and taking the limit as $n \to \infty$, we see that (7.3) is valid also for $F - u$. □

Finally,

Theorem 7.8. *If we extend u by defining it to be f on the boundary, then this extension is continuous.*

Proof: Let $z \in \Omega$ be an interior point, and let $\delta(z)$ be the distance from z to the boundary. Then, using Cauchy Schwarz, and Lemma 7.6, we show (the details appear later)

$$\begin{aligned} &|\,\mathrm{Av}(F)(z) - \mathrm{Av}(u)(z)|^2 \\ &\qquad \leq \frac{16}{\pi} \int_{B(z,2\delta)\cap\Omega} |\nabla F(w)|^2 + |\nabla u(w)|^2 \, dw. \end{aligned} \tag{7.4}$$

Now, as z approaches y, then $\delta \to 0$, and the integral above tends to zero.

This follows from the following

Exercise: show that if $g \in L^2(E)$, that if z_n is a convergent sequence of points in E and if $\delta_n > 0$ tends to zero as $n \to \infty$, then

$$\lim_{n\to\infty} \int_{B(z_n,\delta_n)} |g(x)|^2 \, dx \to 0.$$

Thus, we have from (7.4)

$$|\operatorname{Av}(F)(z) - \operatorname{Av}(u)(z)| \to 0 \text{ as } z \to y.$$

However, as u is harmonic, $\operatorname{Av}(u)(z) = u(z)$, while since F is continuous, $\operatorname{Av}(F)(z) \to F(y) = f(y)$. Thus we see that $u(z) \to f(y)$ as $z \to y$ □

The final thing to do is to take arbitrary continuous boundary data f (not necessarily given by the restriction to $\partial\Omega$ of F).

Theorem 7.9. *Assume that Ω is convex. Let $f : \partial\Omega \to \mathbb{R}$ be continuous. Then there exists a harmonic function $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ with boundary values coinciding with f.*

• Much more is true than what we've described here. In any dimension, if Ω satisfies an 'exterior cone condition', then there is a solution to the Dirichlet problem for arbitrary continuous boundary data f. We will not pursue this question any further in this course, however.

7.6 Deferred proofs

Proof of Proposition 7.4: Let

$$B(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(\theta) d\sigma(\theta).$$

Then we compute $d/drB(r)$ and show that it is zero:

$$\begin{aligned}
\frac{d}{dr} B(r) &= \frac{1}{|\partial B(x,1)|} \frac{d}{dr} \int_{\partial B(0,1)} u(x+ry) d\sigma(y) \\
&= \int_{\partial B(0,1)} d_n u(x+ry) d\sigma(y) \\
&= r \int_{B(0,1)} (\Delta u)(x+ry)\, dy = 0.
\end{aligned}$$

So $B(r)$ is constant in r. But the limit as $r \to 0$ is $u(x)$. This proves the equality of the first and third terms. The remaining equality comes from integrating in r.

Conversely, if $u \in C^2(\Omega)$ satisfies the mean value property, then Δu integrated over any ball is zero, showing that Δu is pointwise zero. □

Proof of Theorem 7.3: To prove the theorem, we use properties of convolutions. Let Ω_ϵ be the set of points in Ω distance at least ϵ from the boundary. Let $\phi(x)$ be a bump function that is supported in $B(0,1)$, has integral 1 and is also radially symmetric, i.e. $\phi(x) = \tilde{\phi}(|x|)$. I claim that if u is weakly harmonic, then $u * \phi_r$ is weakly harmonic in Ω_ϵ for every $r < \epsilon$. To see this, we compute for $\psi \in C_c^\infty(\Omega_\epsilon)$

$$\begin{aligned}\int_\Omega (u * \phi_r)\Delta\psi &= \int\int u(x - ry)\phi(y)\Delta\psi(x)\,dy\,dx \\ &= \int \phi(y)\int u(x - ry)\Delta\psi(x)\,dx\,dy\end{aligned}$$

which vanishes since u is weakly harmonic. But $u * \phi_r$ is smooth: therefore it is harmonic in Ω_ϵ.

Now we exploit the mean value property. Since $u_{r_1} = u * \phi_{r_1}$ is harmonic in Ω_ϵ, for $r_1 < \epsilon$, it satisfies the mean value property, and therefore, we have for $x \in \Omega_\epsilon$ and $r_1 + r_2 < \epsilon$,

$$\begin{aligned}(u_{r_1} * \phi_{r_2})(x) &= \int_{B(0,1)} u_{r_1}(x + r_2 y)\phi(y)\,dy \\ &= \int_0^1 ds s^{n-1}\int_{\partial B(0,1)} u_{r_1}(x + r_2 s\omega)\phi(s\omega)\,d\omega \\ &= \int_0^1 s^{n-1}\tilde{\phi}(s)\int_{\partial B(0,1)} u_{r_1}(x + rs\omega)\,d\omega\,ds \\ &= u_{r_1}(x)\int_0^1 s^{n-1}\tilde{\phi}(s)\,ds = u_{r_1}(x).\end{aligned}$$

Thus, $(u * \phi_{r_1}) * \phi_{r_2} = u_{r_1}$ in Ω_ϵ. However, we also have $(u * \phi_{r_1}) * \phi_{r_2} = (u * \phi_{r_2}) * \phi_{r_1} = u_{r_2}$. So $u_{r_1} = u_{r_2}$ in Ω_ϵ. Now we take the limit $r_1 \to 0$ keeping r_2 fixed, and we find that $u = u_{r_2}$ a.e. in Ω_ϵ, since by Theorem 2.1 of Chapter 3 of the text, $u * \phi_{r_1}$ converges to $u(x)$ a.e. as $r_1 \to 0$. Thus, we correct u on a set of measure zero in Ω_ϵ to make it equal to u_{r_2} there, and we have shown that it becomes harmonic. Taking a sequence of ϵ's tending to zero completes the argument. □

Proof of Lemma 7.5: We note that for all $g \in C_c^\infty(\Omega)$, we have

$$\int_\Omega u(x)\nabla_i g(x) = -\int_\Omega \nabla_i u(x) g(x)\,dx.$$

Now recall that $u - F$ is the limit of a sequence of functions $\phi_n \in S_0$ in the norm of $\mathcal{H}$. In particular, the functions $\nabla_i\phi_n$ are Cauchy for each i in L^2, and therefore converge in $L^2(\Omega)$ to a function v_i. Also, by Proposition 7.1, the ϕ_n are Cauchy in L^2, and therefore converge to $u - F$ in L^2. Hence we have, for any $g \in C_c^\infty(\Omega)$,

$$\int_\Omega (u(x) - F(x))\nabla_i g(x) = \lim_{n\to\infty} \int_\Omega \phi_n(x)\nabla_i g(x) =$$
$$- \lim_{n\to\infty} \int_\Omega \nabla_i \phi_n(x) g(x) = - \int_\Omega v_i g(x).$$

On the other hand, integrating by parts directly we have

$$\int_\Omega (u(x) - F(x))\nabla_i g(x) = - \int_\Omega \nabla_i (u(x) - F(x)) g(x).$$

As this is true for all $g \in C_c^\infty(\Omega)$, the two identities above imply that $\nabla_i(u-F) = v_i$ for each i. And since each v_i is in L^2, we see that $\nabla(u-F) \in L^2(\Omega)$. As $\nabla F \in L^2(\Omega)$, we find that $\nabla u \in L^2(\Omega)$. □

Proof of Lemma 7.6: Let $y \in \partial\Omega$ be a closest point on the boundary to x. Then the hyperplane through y normal to the line xy is a supporting hyperplane for Ω. Choose coordinates so that y is the origin and x is the point $(0, \delta e_2)$.

We use Proposition 7.1. Let $\partial\Omega$ be given near by $z_2 = k(z_1)$, where $z = (z_1, z_2)$ is a general point in $\mathbb{R}^2$. For $|z_1| \le \delta/2$, let I_{z_1} be the interval from $k(z_1)$ to $\delta + \sqrt{(\delta/2)^2 - |z_1|^2}$ which is the 'upper' boundary of the ball $B(x, \delta/2)$. Applying Proposition 7.1 on the interval I_{z_1} we get

$$\int_{I_{z_1}} |v(z_1,t)|^2 \, dt \le |I_{z_1}|^2 \int_{I_{z_1}} |\nabla_2 v(z_1,t)|^2 \, dt.$$

Estimating $|I_{z_1}| \le 2\delta$, we integrate over the ball of radius $\delta/2$ in z_1 to get

$$\int dz_1 \int_{I_{z_1}} |v(z_1,t)|^2 \, dt \le 4\delta^2 \int dz_1 \int_{I_{z_1}} |\nabla_{z_n} v(z_1,t)|^2 \, dt.$$

The integral on the left is bounded below by the LHS of (7.3), while the RHS is bounded above by the RHS of (7.3). This establishes (7.3). □

Derivation of Equation (7.4):

$$\begin{aligned}
&|\operatorname{Av}(F)(z) - \operatorname{Av}(u)(z)|^2 \\
&\quad = \frac{1}{|B(z,\delta/2)|^2} \Big| \int_{B(z,\delta/2)} (F(w) - u(w)) \, dw \Big|^2 \\
&\quad \le \frac{1}{|B(z,\delta/2)|^2} \int\limits_{B(z,\delta/2)} dw \times \int\limits_{B(z,\delta/2)} |F(w) - u(w)|^2 \, dw \\
&\quad \le \frac{16}{\pi} \int_{B(z,2\delta)\cap\Omega} |\nabla F(w) - \nabla u(w)|^2 \, dw
\end{aligned}$$

where we used (7.3) in the last step. (Notice that this step would not work in higher dimensions, as we would get negative powers of δ in the last step.) □

Proof of Theorem 7.9: I will only sketch the proof for $n = 2$. The text has more details.

We use a density argument. Thus, we first extend f to a continuous function $\tilde{f}$ on a ball B containing Ω. The text has a proof that this is possible. It involves use of the Urysohn Lemma, and a limiting argument. Then we find a sequence of C^1 functions F_n on B converging uniformly to $\tilde{f}$. This is possible using, say, the density of polynomials in $C(B)$. For each F_n we solve the Dirichlet problem as above, obtaining a harmonic function u_n. Clearly, u_n converges uniformly on the boundary to f.

I claim that u_n converges uniformly on $\overline{\Omega}$ to a harmonic function. This follows from the following properties of harmonic functions:

Lemma 7.10. *(i) (Maximum Principle) Let $u \in C(\overline{\Omega})$ be harmonic in Ω. Then the maximum and minimum values of u occur on the boundary.*

(ii) Let (u_n) be a sequence of harmonic functions converging uniformly on Ω. Then the limit function is also harmonic.

Proof: The mean value property. □

Part (i) of the lemma shows that the sequence (u_n) converges uniformly on Ω, and part (ii) finishes the proof. □