Math 3325, 2017

Problem Set 2

Discuss in tutorial on August 7 and 14

These questions are all about the problem of finding the closest point in $L^p([0,1])$, in the subspace

$$S_p = \{f \in L^p([0,1]) \mid \int_0^1 x f(x)\, dx = 0\},$$

to the function $g(x) = 1$.

1. Show that S_p is a closed subspace of $L^p([0,1])$, for all $p \in [1, \infty)$.

2. For $p = 2$, find the closest point in S_2 to g (in the L^2 metric).

3. For $p = 1$, show that for every $\epsilon > 0$ there is an $f \in S_1$ such that $\|g - f\|_1 \leq 1/2 + \epsilon$, but that there is no $f \in S_1$ such that $\|g - f\|_1 \leq 1/2$. In particular, there is no closest point to g in S_1 in the L^1 metric.

4. What happens in L^p for $1 < p < 2$? Hint: if $g = f + h$, where $f \in S_p$, use Hölder's inequality

$$\left| \int_0^1 u(x) v(x)\, dx \right| \leq \|u\|_{L^p([0,1])} \|v\|_{L^q([0,1])}, \quad p^{-1} + q^{-1} = 1,$$

to get a lower bound on $\|h\|_p$. Then use the fact that equality in Hölder's inequality will occur if $u, v \geq 0$ and $u = c v^{q-1}$.